

From Begriffsschrift to Grundgesetze

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My aim here is to describe some differences between the *Begriffsschrift* presented by Frege in his *Begriffsschrift* of 1879 and that presented by him in his *Grundgesetze* of 1893.

There is less clarity and consistency in *Begriffsschrift* than in *Grundgesetze* about how the symbols of Frege's system are to be interpreted. In *Begriffsschrift* Frege talks about expressions "standing for" or "expressing" or "meaning" various things. To interpret, for example, the expression

$$\vdash A$$

as standing for the judgment that opposite magnetic poles attract each other, we would take "A" to stand for the *thought* that opposite magnetic poles attract each other. Then

$$\dashv A$$

would stand for what Frege calls the *conceptual content* of "A" (or *content*, for short), and

$$\vdash A$$

would stand for the *judgment* that that content "obtains", "is affirmed", or "takes place". The notions *thought*, *content* and *judgment* are far from clear, but only in the case of the second does Frege try to make them more precise. The content of the sentence "The Greeks defeated the Persians at Plataea" is intended to be whatever the thought that it expresses has in common with the thought that "The Persians were defeated by the Greeks at Plataea" expresses. In the first sentence "The Greeks" acts as grammatical subject and "defeated the Persians at Plataea" acts as grammatical predicate, whereas in the second "The Persians" acts as grammatical subject and "were defeated by the Greeks at Plataea" acts as grammatical predicate. This imparts some difference to the thought that they express, but one that Frege took to be unimportant for "ideography" because "the consequences derivable from the first, when it is combined with certain other judgments, always follow also from the second, when it is combined with these same judgments." This is not to say what the content of sentences like these *is*, only the conditions under which they are the same. But this is the best we get from Frege. To express the judgment that the Queen of England is wise, we might use

$$\vdash \Phi(A)$$

by taking "A" to stand for the Queen of England, and " Φ " to stand for the property of being wise. Then " $\Phi(A)$ " would stand for the thought that the Queen of England is wise, " $\dashv \Phi(A)$ " would stand for the content of " $\Phi(A)$ ", and " $\vdash \Phi(A)$ " would stand for the judgment that that content is the case. Note that in this case "A" does not stand for a thought but names an object (the Queen of England), and its content is that object. Since an object cannot be an object of judgment,

$$\vdash A$$

cannot be interpreted - it is an ill-formed expression. Thus not only is there a fundamental difference between the kinds of thing that the symbol “A” can stand for, but there is the less than elegant need to restrict it to having only judgable content if it is to be used in expressions like this last one.

Even less appealingly there is, in *Begriffsschrift*, the need to interpret the symbol “A” in different ways in the context of different expressions. This was forced upon Frege by his chosen solution to a problem about the expression of identity claims. We’d like to be able to express, for example, the judgment that the morning star is identical with the evening star. We might think that we could do so by using the expression

$$\vdash(A=B)$$

and taking the content of “A” (what it stands for) to be the morning star and the content of “B” to be the evening star. But according to the account of content given by Frege in *Begriffsschrift*, there is no relevant difference between expressions that have the same content. Since the evening star *is* the morning star, the contents of “A” and “B” are the same, and so we ought then be able to express the same judgment using

$$\vdash(A=A).$$

But these seem to express quite different judgments, because the first is informative in a way that the second is not. So it seems, and Frege concluded, that the content of “A” in the first expression cannot be the morning star. Nor can it be itself - the symbol “A” - for then to be consistent the content of “B” would have to be “B”, and then the expression would stand for the wrong judgment: that the symbol “A” is identical to the symbol “B” (an impossible judgment, for Frege, because judgment for him is factive). Frege’s solution was to interpret the thought that the morning star is the evening star to be about the expressions “the morning star” and “evening star”, not as the thought that those expressions are identical, but that they denote the same thing. Thus, by taking the content of “A” to be the morning star and the content of “B” to be the evening star, and by introducing the tri-bar “ \equiv ” as a way of expressing that two expressions have the same content, we are able to express the intended judgment as

$$\vdash(A\equiv B).$$

This solves the problem, but at a double cost. First, it means that expressions in the system must be interpreted differently in different contexts. If “A” occurs adjacent to a tri-bar, as in this last expression, then it must be interpreted as standing for itself, the symbol “A”, otherwise it must be interpreted as standing for its content (in this case, the morning star). Second, the proposed interpretation of the original judgment seems to be just plain wrong - intuitively it is *not* a judgment about symbols and what they stand for, but about the morning star and the evening star. Frege allows the tri-bar to be used between expressions standing not just for objects but for thoughts as well. Thus, if “A” stands for the thought that opposite magnetic poles attract each other, and “B” stands for the thought that if magnetic poles are opposite then they attract each other, then his system allows us to use the expression

$$\vdash(A\equiv B)$$

to stand for the judgment that the sign “A” and the sign “B” have the same conceptual content. Here “A” must be interpreted as standing for itself, a symbol, and not for the thought that it expresses, as it must be interpreted in the expression

$$\neg A.$$

This makes quite a difference to the judgment expressed. There must be a difference, because we can make the judgment expressed by the first without knowing whether or not opposite magnetic poles do in fact attract each other, but we can’t so make the judgement expressed by the second.

As for the interpretation of other expressions, Frege says in *Begriffsschrift* that

$$\begin{array}{l} \neg A \\ \quad \neg B \end{array}$$

stands for the judgment that “[the situation in which A is denied and B is affirmed] does not take place, but [one in which A is affirmed or B is denied] does”; that

$$\neg \neg A$$

means that “A does not take place”; that

$$\begin{array}{l} \neg \neg A \\ \quad \neg B \end{array}$$

means that “A and B cannot both be denied” or that “no possibility other than A and B is thinkable”; and that

$$\begin{array}{l} \neg \neg \neg A \\ \quad \neg B \end{array}$$

means that “the case in which both A and B are affirmed occurs.” While we can understand what Frege is trying to say by talking about A and B being denied, affirmed, taking place, occurring, and being thinkable, there is a lack of precision and clarity here that stands in need of fixing.

In the *Begriffsschrift* of *Grundgesetze* we find some substantial changes to the semantics of the system, and a much clearer account of it from Frege. Three developments are largely responsible for this: (i) the drawing of a sharper distinction between function and object, (ii) the interpretation of (nearly) all *Begriffsschrift* expressions as names of functions and objects, and (iii) the division of the content of an expression into sense and denotation.

Consider again the expression

$$\neg A.$$

According to *Begriffsschrift*, “A” stands for a thought (if the expression is to have an interpretation), “—A” stands for the content of “A”, and “┌—A” stands for the judgment that that content is the case. According to *Grundgesetze*, “A” is the name of an object, “—A” is the name the value of the truth-valued function, —(ξ), when its argument is the object named by “A”, and “┌—A” expresses the judgment that that value is the True. “A” can be the name of any object, among which Frege now includes the two truth-values *the True* and *the False*, and it need only purport to name. Thus it might name or purport to name a person (John Howard, Santa Claus, or the queen of England, for example), or a truth value (the truth value of the proposition that grass is green, or that Santa Claus lives at the North Pole, for example). Note that whereas in *Begriffsschrift* “A” could only stand for a proposition and not for an object, here “A” can stand for a proposition (by being a name for its truth-value), and also for any object (by being a name for that object). The truth-valued function, —(ξ), is a function from objects to truth-values, and is defined to be the True if its argument is the True, and the False in every other case. Thus it is defined for every object, even objects like the Queen of England that are not themselves truth-values. By making this function a total function, Frege removed the need to place any restriction upon the kinds of objects that “A” can name or purport to name. The expression “┌—A”, like all expressions of the form “┌—Φ”, is one of the few expressions in *Grundgesetze* that are not now regarded as being names. These expressions are not names, but stand for the judgment that —Φ is the True. Judgments cannot be named, because they are not objects.

The negation-stroke and the condition-stroke are also interpreted in *Grundgesetze* as being names of functions. The expression

$$\top \xi$$

names that function from objects to truth-values which takes the value the False if its ξ-argument is the True, and the value the True in all other cases. It follows that “┌Δ”, “┌—Δ”, “—┌Δ” and “—┌ —Δ” always name the same object, and that we can move between them by “amalgamating horizontals”. The expression

$$\begin{array}{c} \top \xi \\ \text{┌} \xi \end{array}$$

names that function from objects to truth-values which takes the value the False if its ζ-argument is the True and its ξ-argument is any object other than the True, and the value the True in all other cases. It follows that

$$\begin{array}{c} \top \Gamma \\ \text{┌} \Delta \end{array} \quad \text{and} \quad \begin{array}{c} \text{—} \top (\text{—} \Gamma) \\ \text{┌} (\text{—} \Delta) \end{array}$$

always name the same object, and that we can move between them also by amalgamating horizontals. Thinking of the negation- and condition-strokes as truth-valued function names in this way allows for a much neater interpretation of

$$\begin{array}{c} \text{┌} \text{—} \text{—} \text{—} \text{A} \\ \text{—} \text{—} \text{—} \text{B} \end{array}$$

We can say that this expresses the judgment that

$$\frac{}{\frac{}{\top}A}$$

names the True, which will so just in case either “A” names the True or “B” names the True. This is much crisper than saying that it means “no possibility other than A and B is thinkable”, as Frege says in *Begriffsschrift*.

Drawing a distinction between the sense and denotation of an expression allowed Frege to express identity claims in the Begriffsschrift of *Grundgesetze* in a much more satisfying way. As we have seen, to express the judgment that the morning star is the evening star in the system of *Begriffsschrift*, we would use the expression

$$\frac{}{\top}(A\equiv B),$$

where “A” and “B” are taken in other contexts to stand for the morning star and evening star respectively, but in this context are taken to stand for themselves. In the system of *Grundgesetze* we no longer have the tri-bar and would use, instead, the expression

$$\frac{}{\top}(A=B)$$

and take “A” and “B” to stand for the morning and evening stars as they do in all other contexts. Frege’s new account of why this last expression expresses a more informative (and, therefore, seemingly different) judgment than does the expression

$$\frac{}{\top}(A=A)$$

is that the expressions “A” and “B”, while having the same denotation, differ in sense, and it is the difference in sense that accounts for the differing informativeness of the two judgments. Thus there is no need to interpret, unsatisfyingly, the expressions “A” and “B” differently in different contexts, and there is no need to interpret, implausibly, the judgment of identity as being about symbols and what they stand for. The sense of an expression has to do with the manner in which we think about its denotation, but, for Frege, this is not the concern of Logic, which is concerned with expressions just at the level of denotation. Thus, whereas the conceptual content of an expression in *Begriffsschrift* was a tangle of what Frege now recognised as its sense and denotation, in *Grundgesetze* the conceptual content is taken to be its denotation. Frege’s achievement in distinguishing the sense of an expression from its denotation was not to be able to incorporate the former into his system, but rather to leave it out.

As I have mentioned, in the Begriffsschrift of *Grundgesetze* Frege takes (nearly) all symbols to be names that denote (or purport to denote) either an object or a function, and that this allows a more simple and precise interpretation of the expressions of the system. I want, finally, to make some comments about the notion of function that Frege develops. Functions are sharply distinguished from objects, standing “in need of completion” in a way that objects are not. But if they are not objects then how can they be named? How can an expression denote a function, as Frege says it can, if there is no *thing* to be denoted? On Frege’s account, the expressions “the morning star” and “the

evening star” are names of objects with different senses but the same denotation, the last fact being expressible in Begriffsschrift notation by the expression

$$\vdash (A=B),$$

taking “A” to be an abbreviation of “the morning star” and “B” to be an abbreviation of “the evening star”. Similarly, the expressions “ $\xi^2 - 4\xi$ ” and “ $\xi(\xi - 4)$ ” are taken by Frege to be names of functions with different senses but the same denotation. How can this last fact be expressed in the language of Begriffsschrift? An obvious first thought is use the expression

$$\vdash \xi^2 - 4\xi = \xi(\xi - 4).$$

But the expression after the horizontal names a single function, and what we have here stands for the judgment that this function takes the value the True for all values of its argument. This is a true consequence of the identity we are trying to express, but it is only a consequence of it, not the identity itself. On second thought we might try

$$\vdash \xi^2 - 4\xi = \zeta(\zeta - 4).$$

But the expression after the horizontal still names a single function. Worse still, it names a function of two variables that does not take the value the True for all values of its ξ - and ζ - arguments, and so we have here an expression of an impossible judgment. A final thought might be to use Frege’s *course-of-values* notation and the expression

$$\vdash \varepsilon(\varepsilon^2 - 4\varepsilon) = \alpha(\alpha(\alpha - 4)).$$

But the course of values of a function is taken by Frege to be an object and so can’t be the function itself. So what we have expressed here is not the judgment that two functions are identical but that their course of values are identical. This, again, is a true consequence of the identity we are trying to express, but it is only a consequence of it, not the identity itself.

The solution to the problem of expressing the identity of two functions is, I think, to recognise that it cannot be done. Functions are not objects, identity is a relation that can only hold between objects, so functions cannot be identified. But it still makes sense, in a way, to talk about an expression denoting a function, and about two such function-names having the same denotation. An expression denotes a function (is a function-name) not in virtue of denoting an object, but in virtue of denoting an object when completed by any name that denotes an object.¹ Two function-names have the same denotation if and only if for any name that denotes an object the function-names denote the same object when completed by that name. If this is what Frege had in mind, then I think it’s misleading to talk of some expressions as denoting objects and of others as denoting functions, because it leads us to think that functions are *things* that can be denoted. It would, I think, be less misleading to talk of some expressions as having *objectual* denotation and of others as having *functional* denotation. There is no difference in what it *is* that these expressions denote, just in the way that they do it.²

¹ Frege says something to this effect in GG2 p. 84.

² My discussion here has been influenced by what Furth says about the matter. See his introduction to GG2, and his ???? paper.

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