Lattice Paths and Catalan Numbers

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This article was inspired by a problem appearing in the February 1990 issue of *Crux Mathematicorum* and brought to our attention by Bill Sands [3]. Our aim was to find a *nice* answer with a *nice proof* of the following problem.

Problem. How many walks of n one-unit steps, say **N**, **E**, **W** or **S** are there which begin at the origin and never go to the left of the y-axis?

The nice answer, $\binom{2n+1}{n}$, emerged almost immediately, but a nice proof was more difficult to find.

Though much of what follows is not new, the richness of the problem, its many ramifications and the many possible solutions has prompted us to make our results available to a wider audience in the hope that it will be as fascinating to them as it has been to us.

§1. Linear Walks and Catalan Numbers

A *linear n-walk* on the x-axis we will mean a connected path of n steps, labelled **L** or **R** (for *left* and *right*), such that,

* the path starts at the origin;

* each step is a one unit line segment parallel to the x-axis.

Paths which never move left of the origin, i.e. never reach $x = -1$, will be called *half-linear n-walks*.

Notation. Let $a_{n,k}$ denote the number of linear n-walks which end at the point $x = k$.

The number of half-linear n-walks which end at $x = k$, for $k \geq 0$, will be denoted by $b_{n,k}$.

The total number linear n-walks will be a_n and the total number of half-linear n-walks will be denoted by b_n.

Clearly there are a total of $a_n = 2^n$ linear n-walks.

It is also easy to find a formula for $a_{n,k}$, since each linear n-walk ending in $x = k$ has, say, a **L**'s and b **R**'s, where a and b satisfy the equations, $a + b = n$ and $a - b = k$. Solving these equations yields:

$$a = \frac{n+k}{2}, \qquad b = \frac{n-k}{2}$$

Hence $a_{n,k} = 0$ if $n+k$ is not even, and the linear n-walks ending at $x = k$ are completely determined by the positions of the a **L**'s on the path. Hence we have:

$$a_{n,k} = \begin{cases} \binom{n}{\frac{1}{2}(n+k)} & \text{if } n+k \text{ is even;} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note. We also observe that each path counted by $a_{n+1,k}$ is obtained by adding a step **R** to a path of length n ending in $x = k-1$, or a step **L** to a path of length n ending at $x = k+1$. Hence we have the recurrence formula,

$$a_{n+1,k} = a_{n,k-1} + a_{n,k+1} \tag{2}$$

which, together with the initial conditions $a_{0,0} = 1$, $a_{0,k} = 0$ for $k \neq 1$, yields the following Pascal's triangle:

			$a_{0,0}$			
		$a_{1,-1}$	$a_{1,0}$	$a_{1,1}$		
	$a_{2,-2}$	$a_{2,-1}$	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	
$a_{3,-3}$	$a_{3,-2}$	$a_{3,-1}$	$a_{3,0}$	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

			1			
		1	0	1		
	1	0	2	0	1	
1	0	3	0	3	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1. The $a_{n,k}$'s.

Note. We could also deduce formula (1) from the recurrence equation (2) by applying a simple induction.

Half-linear walks

For $k \geq 0$ we have,

$$b_{n,k} = a_{n,k} - c_{n,k}, \tag{3}$$

where $c_{n,k}$ is the number of linear n-walks which intersect $x = -1$ and end at $x = k$.

We will show that, for $k \geq 0$,

$$c_{n,k} = a_{n,k+2}, \tag{4}$$

by using a standard reflection argument (see for example, Cohen [1, p144]). That is, we set up a bijection between the set S of linear n-walks starting at $x = 0$ which intersect $x = -1$ and end at $x = k$, and the set T of linear n-walks which start at $x = -2$ and end at $x = k$. The bijection $S \longrightarrow T$ is as follows.

- ⋆ Given a walk in S, we find the first step which intersects $x = -1$. Now relabel this step and all previous ones according to the rule $\mathbf{R} \longleftrightarrow \mathbf{L}$. This yields a new string with one more **R** and one less **L**, for a net gain of two **R**'s, i.e. a linear n-walk from the origin to $x = k+2$, or equivalently, a linear n-walk from $x = -2$ to $x = k$.
- ⋆ Conversely, any walk in T must pass through $x = -1$ in order to reach $x = k \geq 0$. Find the first step where the walk intersects $x = -1$ and relabel this step and all previous

§2. Planar Walks

By a *planar n-walk* in the xy-plane we will mean a connected path of n steps, labelled N, E, W or S (for *north, east, west* and *south*), such that,

* the path starts at the origin;
* each step is a one unit line segment parallel to the x or y axis.

A *half-planar n-walk* will be a planar n-walk which is never left of the y-axis, i.e. never intersects the line $x = -1$.

Notation. Write $a_{n,(k_1,k_2)}$ for the number of planar n-walks which end at the point (k_1, k_2). These are paths of length n, in the lattice, $\mathbb{Z}^2$, of integer points in the xy-plane, which start at the origin, end at the point (k_1, k_2). We will write $b_{n,(k_1,k_2)}$ for the number of half-planar n-walks which end at the point (k_1, k_2) for $k_1 \geq 0$.

The number of planar (respectively half-planar) n-walks which end on the line $x = k$ will be $a_{n,(k,.)}$ (respectively $b_{n,(k,.)}$).

The total number of planar (respectively half-planar) n-walks will be denoted by A_n (respectively B_n)

Note. Clearly there are a total of $A_n = 4^n$ planar n-walks.

Our immediate problem is to determine the number B_n of half-planar n-walks.

A crucial observation

Observe that there is a one-to-one correspondence between the set of planar n-walks and the set of linear $2n$-walks given by:

$$\begin{cases} \mathbf{E} \longleftrightarrow \mathbf{RR} \\ \mathbf{W} \longleftrightarrow \mathbf{LL} \\ \mathbf{N} \longleftrightarrow \mathbf{LR} \\ \mathbf{S} \longleftrightarrow \mathbf{RL} \end{cases} \tag{8}$$

Example 3. The planar 6-walk **NNESWW** corresponds to the linear 12-walk

LRLRRRRLLLLL.

Note. Under this bijection, it's clear that planar n-walks ending on the line $x = k$ correspond to linear $2n$-walks ending at the point $x = 2k$. Hence we have:

> The number of planar n-walks ending on the line $x = k$ is by (1),
>
> $$a_{n,(k,.)} = \binom{2n}{\frac{1}{2}(2n+2k)} = \binom{2n}{n+k}. \tag{9}$$

ones according to the same rule as above. This yields a path from $x = 0$ which intersects $x = -1$ and ends at $x = k$, i.e. a walk in S.

These two correspondences are clearly inverse, hence we have a bijection.

Example 1. For example, **RLRLLLRRR** is a linear 9-walk ending at $x = 1$. It intersects $x = -1$ first at the fifth step, and so corresponds to **LRLRRLRRR**, which can be viewed as a linear 9-walk ending at $x = 1$ but starting at $x = -2$.

To summarize, we have:

The number of half-linear n-walks ending at the point $x = k$, for $k \geq 0$, is given by,

$$b_{n,k} = a_{n,k} - a_{n,k+2} \tag{5}$$

Note. From (5), the $b_{n,k}$ clearly satisfy the recurrence formula,

$$b_{n+1,k} = b_{n,k-1} + b_{n,k+1}, \tag{6}$$

with initial conditions $b_{0,0} = 1$, $b_{0,k} = 0$ for $k \geq 1$ and boundary conditions $b_{n,-1} = 0$ for all n.

Example 2. If $n = 2m$ and $k = 0$ then the half-linear walks end at the origin and so have the same number of **L**'s and **R**'s. They are just the **UD**-sequences of Cohen [1, p.144]. In this case,

$$\begin{aligned} b_{2m,0} &= a_{2m,0} - a_{2m,2} \\ &= \binom{2m}{m} - \binom{2m}{m+1} \\ &= \frac{1}{m+1}\binom{2m}{m}. \end{aligned}$$

Hence we obtain the well-known result that $b_{2m,0}$ is the mth *Catalan number* C_m.

Formulas (1) and (5) allow us to easily calculate the total number, b_n, of half-linear n-walks. Thus,

The total number of half-linear n-walks is given by,

$$\begin{aligned} b_n &= b_{n,0} + b_{n,1} + \cdots + b_{n,n} \\ &= (a_{n,0} - a_{n,2}) + (a_{n,1} - a_{n,3}) + \cdots + (a_{n,n} - a_{n,n+2}) \\ &= a_{n,0} + a_{n,1} \\ &= \begin{cases} \binom{n}{\frac{1}{2}n} & \text{if } n \text{ is even;} \\ \binom{n}{\frac{1}{2}(n+1)} & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{7}$$

Example 4. As a check we see that

$$\sum_{k=-n}^{n} \binom{2n}{n+k} = 4^n = A_n.$$

Question. What happens to half-planar n-walks under this bijection?

They don't necessarily correspond to half-linear $2n$-walks since we may start with a step labelled **N**. However, each **N** and **S** step contributes an equal number of **L**'s and **R**'s under bijection (8). Moreover, given a half-planar n-walk, at each step there will be at least as many **E**'s as **W**'s preceeding it, i.e. at least as many **R**'s as **L**'s are contributed by the **E**, **W** steps under (8). Hence, given a half-planar n-walk, we obtain a half-linear $(2n+1)$-walk if we,

- ⋆ perform bijection (8) to obtain a linear $2n$-walk;
- ⋆ add **R** to the beginning of the sequence.

Conversely, every half-linear $(2n+1)$-walk begins with a step labelled **R**. Hence we obtain a half-planar n-walk we just reverse the above procedure, i.e.

- ⋆ delete the beginning **R**;
- ⋆ pair the remaining **L**'s and **R**'s in the obvious way;
- ⋆ apply bijection (8).

Note. The above correspondences thus yield a one-to-one correspondence between half-planar n-walks and half-linear $(2n+1)$-walks.

In particular, we have by (7), the answer to our original problem:

The total number of half-planar n-walks is

$$B_n = \binom{2n+1}{\frac{1}{2}(2n+1+1)} = \binom{2n+1}{n+1} = \binom{2n+1}{n}. \tag{10}$$

Note. Under the above correspondence, each half-planar n-walk ending on the line $x = k$ corresponds to a half-linear $(2n+1)$-walk ending at the point $x = 2k+1$. Hence by (5) and (1) we have:

The number of half-planar n-walks ending on the line $x = k$ is given by,

$$b_{n,(k,.)} = b_{2n+1,2k+1} = \binom{2n+1}{n+k+1} - \binom{2n+1}{n+k+2} = \binom{2n}{n+k} - \binom{2n}{n+k+2}. \tag{11}$$

Example 5. The walk **NEENW** is half-planar of length 5 ending on the line $x = 1$. This corresponds to the half-linear 11-walk **RLRRRRRLRLL** which finishes at $x = 3$.

Remark. Substituting $k = 0$ in (11), we obtain the following further connection with the Catalan numbers.

The number of half-planar n-walks which end on the y-axis is

$$\begin{aligned} b_{n,(0,.)} = b_{2n+1,1} &= \binom{2n+1}{n+1} - \binom{2n+1}{n+2} \\ &= \frac{1}{n+2}\binom{2n+2}{n+1} \qquad (12) \\ &= C_{n+1}, \end{aligned}$$

(the $(n+1)$-th Catalan number).

Comment. It is easy to establish (12) by constructing the following bijection between the set of half-planar n-walks and the set of strings of $n+1$ pairs (balanced, or nested) bracketings. For each walk, we replace **E** by '((', **N** by '()', **S** by ')(' and **W** by '))', then add one more pair of brackets, a left bracket '(' at the beginning and a right bracket ')' at the end. (The final pair of brackets is needed in case the walk starts with an **S**.) This process is clearly reversible, hence we have a bijection.

Example 6. **ENEWSW** corresponds to (((()(()))())).

By (12) there are C_{n+1} half-planar n-walks which end on the y-axis. However, $2^{n-2k}C_k$ of these have exactly k pairs of E's and W's. Hence we have the following identity,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k}\binom{n}{2k} C_k = C_{n+1}. \qquad (13)$$

Or, alternatively,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k+1} 2^{n-2k}\binom{n}{2k}\binom{2k}{k} = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

Planar n-walks to a point

We conclude this section by finding formulas for $a_{n,(k_1,k_2)}$ and $b_{n,(k_1,k_2)}$. First it is instructive to display some of these for small values of n. Each table is centered at the origin and the number above each point is the number $a_{n,(k_1,k_2)}$ corresponding to the point (k_1, k_2).

$n = 0$

1

$n = 1$

	1	
1	0	1
	1	

$n = 2$

		1		
	2	0	2	
1	0	4	0	1
	2	0	2	
		1		

$n = 3$

			1			
		3	0	3		
	3	0	9	0	3	
1	0	9	0	9	0	1
	3	0	9	0	3	
		3	0	3		
			1			

$n = 4$

				1				
			4	0	4			
		6	0	16	0	6		
	4	0	24	0	24	0	4	
1	0	16	0	36	0	16	0	1
	4	0	24	0	24	0	4	
		6	0	16	0	6		
			4	0	4			
				1				

Table 2. $a_{n,(k_1,k_2)}$ for $n = 0, 1, 2, 3, 4$.

We can immediately make three observations.

(1) The outer diagonal entries are the binomial coefficients $\binom{n}{k}$.

(2) If we regard the above tables as cross-sections (corresponding to $z = n$) of a square cone defined by the rays $z = x$ and $z = y$ (in the xz- and yz-planes respectively), then the numbers on each layer are the sums of the four adjacent numbers on the previous layer.

(3) The non-zero internal entries of each table are products of the corresponding outer diagonal entries, i.e. each table appears to be a complete multiplication table, governed by the outer diagonals. This suggests that,

$$a_{n,(k_1,k_2)} = \begin{cases} \binom{n}{\frac{1}{2}(n+k_2+k_1)}\binom{n}{\frac{1}{2}(n+k_2-k_1)} & \text{if } n \equiv k_1+k_2 \pmod 2; \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Note. Equation (14) is a direct generalization of the situation in one dimension (Equation (1)).

Our problem now is to prove (1), (2) and (3).

To justify the first observation, notice that the outer diagonal entries count walks generated by only two kinds of steps, **EN** in the first quadrant, **WN** in the second, **WS** in the third, and **ES** in the fourth quadrant. For example, in the first quadrant the outer diagonal is along the line $x+y=n$, the k-th entry is the number $a_{n,(k,n-k)}$ of all planar n-walks with k **E**'s and $n-k$ **N**'s. This number is $\binom{n}{k}$ since once the k **E**-steps are chosen, the **N**-steps are determined.

Next observe that each planar $(n+1)$-walk to (k_1,k_2) arises from a planar n-walk to one of the points (k_1-1,k_2), (k_1+1,k_2), (k_1,k_2-1) or (k_1,k_2+1). Hence the second observation follows from the recurrence,

$$a_{n+1,(k_1,k_2)} = a_{n,(k_1-1,k_2)} + a_{n,(k_1+1,k_2)} + a_{n,(k_1,k_2-1)} + a_{n,(k_1,k_2+1)}. \tag{15}$$

Finally, we deduce (14) from the recurrence formula (15) using induction. It's clearly true for small values of n. Assuming that (14) holds for numbers up to n, we find,

$$\begin{aligned} a_{n+1,(k_1,k_2)} &= \binom{n}{\frac{1}{2}(n+k_2+k_1-1)}\binom{n}{\frac{1}{2}(n+k_2-k_1-1)} \\ &\qquad + \binom{n}{\frac{1}{2}(n+k_2+k_1+1)}\binom{n}{\frac{1}{2}(n+k_2-k_1-1)} \\ &\quad + \binom{n}{\frac{1}{2}(n+k_2+k_1-1)}\binom{n}{\frac{1}{2}(n+k_2-k_1-1)} \\ &\qquad + \binom{n}{\frac{1}{2}(n+k_2+k_1+1)}\binom{n}{\frac{1}{2}(n+k_2-k_1-1)} \\ &= \binom{n}{\frac{1}{2}(n+k_2+k_1-1)}\left[\binom{n}{\frac{1}{2}(n+k_2-k_1+1)} + \binom{n}{\frac{1}{2}(n+k_2-k_1-1)}\right] \\ &\quad + \binom{n}{\frac{1}{2}(n+k_2+k_1+1)}\left[\binom{n}{\frac{1}{2}(n+k_2-k_1-1)} + \binom{n}{\frac{1}{2}(n+k_2-k_1+1)}\right] \\ &= \binom{n+1}{\frac{1}{2}(n+1+k_2+k_1)}\binom{n+1}{\frac{1}{2}(n+1+k_2-k_1)}. \end{aligned}$$

[Here we use the convention that $\binom{n}{r}=0$ if r in not an integer.]

Alternatively, we can deduce (14) from the correspondence,

$$\begin{cases} \mathbf{E} \longleftrightarrow \mathbf{RR} \\ \mathbf{W} \longleftrightarrow \mathbf{LL} \\ \mathbf{N} \longleftrightarrow \mathbf{LR} \\ \mathbf{S} \longleftrightarrow \mathbf{RL} \end{cases} \tag{$*$}$$

Take a planar n-walk ending at (k_1, k_2), call it $\mathcal{W}$. (For example, **NESSE** finishes at $(2, -1)$.) Define $|\mathcal{W}|_{\mathrm{N}}$ to be the number of N steps in $\mathcal{W}$. Define $|\mathcal{W}|_{\mathrm{S}}$, $|\mathcal{W}|_{\mathrm{E}}$, $|\mathcal{W}|_{\mathrm{W}}$ similarly. Since $\mathcal{W}$ finishes at (k_1, k_2) we have

$$\begin{cases} |\mathcal{W}|_{\mathrm{E}} - |\mathcal{W}|_{\mathrm{W}} = k_1 \\ |\mathcal{W}|_{\mathrm{N}} - |\mathcal{W}|_{\mathrm{S}} = k_2 \end{cases} \qquad (**)$$

Now write $\mathcal{W}$ as a vertical sequence of steps and apply (*) to each step. For example,

	(1)	(2)
N	**L**	**R**
E	**R**	**R**
S	**R**	**L**
S	**R**	**L**
E	**R**	**R**

This produces two new columns, (1) and (2), which can be viewed as linear n-walks.

Because $\mathcal{W}$ ends at (k_1, k_2), conditions $(**)$ mean that linear walk (1) ends at $k_1 - k_2$, linear walk (2) ends at $k_1 + k_2$.

Conversely, *any* linear walk ending at $k_1 - k_2$ can be combined with *any* linear walk ending at $k_1 + k_2$ by using the inverse process to obtain a planar n-walk ending at (k_1, k_2). Thus,

$$\begin{aligned} a_{n,(k_1,k_2)} &= \begin{pmatrix} \text{Number of linear } n\text{-walks} \\ \text{ending at } k_1 - k_2 \end{pmatrix} \times \begin{pmatrix} \text{Number of linear } n\text{-walks} \\ \text{ending at } k_1 + k_2 \end{pmatrix} \\ &= \begin{cases} \binom{n}{\frac{1}{2}(n+k_2+k_1)}\binom{n}{\frac{1}{2}(n+k_2-k_1)} & \text{if } n \equiv k_1 + k_2 \pmod 2; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consider next the numbers $b_{n,(k_1,k_2)}$ which are displayed below for some small values of n.

As in Table 2, each table is centered at the origin. The number above each point is the number $b_{n,(k_1,k_2)}$ corresponding to the point (k_1, k_2).

$n = 0$

1

$n = 1$

1	
0	1
1	

$n = 2$

1		
0	2	
3	0	1
0	2	
1		

$n = 3$

1			
0	3		
6	0	3	
0	8	0	1
6	0	3	
0	3		
1			

$n = 4$

1				
0	4			
10	0	6		
0	20	0	4	
20	0	15	0	1
0	20	0	4	
10	0	6		
0	4			
1				

Table 3. The $b_{n,(k_1,k_2)}$'s for $n = 0, 1, 2, 3, 4$.

These tables can be calculated from Table 2 via the relationship,

$$b_{n,(k_1,k_2)} = a_{n,(k_1,k_2)} - a_{n,(k_1+2,k_2)}, \tag{16}$$

which can be proved by an argument analogous to that which gave formula (5), except this time we reflect about the line $x = -1$.

From (14) we have:

$$b_{n,(k_1,k_2)} = \begin{cases} \binom{n}{\frac{1}{2}(n+k_2+k_1)}\binom{n}{\frac{1}{2}(n+k_2-k_1)} \\ \qquad -\binom{n}{\frac{1}{2}(n+k_2+k_1)+1}\binom{n}{\frac{1}{2}(n+k_2-k_1)-1} & \text{if } n \equiv k_1 + k_2 \pmod 2 \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

§3. Generating Functions

In this section we find the generating functions for the important numbers of the previous sections.

We will write the *generating function* of the sequence $\{a_{n,k}\}$, for fixed n, as,

$$\Phi_n(x) = \sum_{k=-\infty}^{\infty} a_{n,k} x^k. \tag{18}$$

For the first few values of n we have,

$$\Phi_0(x) = 1$$
$$\Phi_1(x) = x^{-1} + x$$
$$\Phi_2(x) = x^{-2} + 2 + x^2 = (x^{-1} + x)^2.$$

Also using recurrence formula (2) we have,

$$\begin{aligned} a_{n+1,k}x^k &= (a_{n,k-1} + a_{n,k+1})x^k \\ &= xa_{n,k-1}x^{k-1} + x^{-1}a_{n,k+1}x^{k+1}. \end{aligned}$$

Hence,

$$\begin{aligned} \Phi_{n+1}(x) &= \sum_{k=-\infty}^{\infty} a_{n+1,k}x^k \\ &= \sum_{k=-\infty}^{\infty} \left[xa_{n,k-1}x^{k-1} + x^{-1}a_{n,k+1}x^{k+1}\right] \\ &= (x + x^{-1})\Phi_n(x). \end{aligned}$$

We thus find that,

$$\Phi_n(x) = (x + x^{-1})^n = x^{-n}(1 + x^2)^n. \tag{19}$$

Comparing coefficients in (19), we find that,

$$a_{n,-n+2i} = \binom{n}{i} \qquad \text{and so} \qquad a_{n,k} = \binom{n}{\frac{1}{2}(n+k)},$$

the formula given in (1).

Next, we want an appropriate generating function, $\Psi_n(x)$, for the $b_{n,k}$'s. It is convenient to extend the definition of the $b_{n,k}$'s to negative values of k via formula (5), i.e.,

$$b_{n,k} = a_{n,k} - a_{n,k+2}.$$

Then it is immediate that

$$\Psi_n(x) = \sum_{k=-\infty}^{\infty} b_{n,k}x^k, \tag{20}$$

satisfies

$$\Psi_n(x) = \Phi_n(x)(1 - x^{-2}) = (x + x^{-1})^n(1 - x^{-2}). \tag{21}$$

Note. In general,

$$\begin{aligned} b_{n,-k} &= a_{n,-k} - a_{n,-k+2} \\ &= a_{n,k} - a_{n,k-2} \\ &= -b_{n,k-2}. \end{aligned} \tag{22}$$

Thus the table of $b_{n,k}$'s is antisymmetric about the $k = -1$ column. In particular, $b_{n,-1} = 0$ for $n = 0, 1, 2, \ldots$, in agreement with our boundary conditions. The first few values are displayed below:

n/k	$\cdots$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	$\cdots$
0	$\cdots$					-1	0	1					$\cdots$
1	$\cdots$				-1	0	0	0	1				$\cdots$
2	$\cdots$			-1	0	-1	0	1	0	1			$\cdots$
3	$\cdots$		-1	0	-2	0	0	0	2	0	1		$\cdots$
4	$\cdots$	-1	0	-3	0	-2	0	2	0	3	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4. The $b_{n,k}$'s including their 'ghost values'.

Note. Identity (6) is now valid for all values of k.

Two dimensions

For fixed n, we have the the following 2-variable generating function for the sequence $\{a_{n,(k_1,k_2)}\}$:

$$\Phi_n(x_1,x_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} a_{n,(k_1,k_2)} x_1^{k_1} x_2^{k_2}. \tag{23}$$

It follows immediately from (15) that,

$$\begin{aligned}\Phi_{n+1}(x_1,x_2) &= \sum_{(k_1,k_2)\in\mathbb{Z}^2} a_{n+1,(k_1,k_2)} x_1^{k_1} x_2^{k_2} \\ &= \sum_{(k_1,k_2)\in\mathbb{Z}^2} x_1 a_{n,(k_1-1,k_2)} x_1^{k_1-1} x_2^{k_2} + x_1^{-1} a_{n,(k_1+1,k_2)} x_1^{k_1+1} x_2^{k_2} \\ &\qquad + x_2 a_{n,(k_1,k_2-1)} x_1^{k_1} x_2^{k_2-1} + x_2^{-1} a_{n,(k_1,k_2+1)} x_1^{k_1} x_2^{k_2+1} \\ &= \left(x_1 + x_1^{-1} + x_2 + x_2^{-1}\right)\Phi_n(x_1,x_2). \end{aligned} \tag{24}$$

Also, we have

$$\begin{aligned}&\Phi_0(x_1,x_2) = 1 \\ &\Phi_1(x_1,x_2) = x_1 + x_1^{-1} + x_2 + x_2^{-1} \quad \text{and so,} \\ &\Phi_n(x_1,x_2) = \left(x_1 + x_1^{-1} + x_2 + x_2^{-1}\right)^n. \end{aligned} \tag{25}$$

Example 7. Comparing coefficients in (25) with formula (14) yields the following multinomial identity,

$$\sum_{d=0}^{n} \binom{n}{\frac{1}{2}(n+k_1-k_2)-d,\ \frac{1}{2}(n-k_1-k_2)-d,\ k_2+d,\ d} = \begin{cases} \binom{n}{\frac{1}{2}(n-k_2-k_1)}\binom{n}{\frac{1}{2}(n+k_1-k_2)} & \text{if } n \equiv k_1+k_2 \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

We use recurrence formula (16) to extend the definition of $b_{n,(k_1,k_2)}$ to all k_1.

The generating function for the sequence, $\{b_{n,(k_1,k_2)}\}$, i.e.,

$$\Psi_n(x_1,x_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} b_{n,(k_1,k_2)} x_1^{k_1} x_2^{k_2}, \tag{26}$$

satisfies,

$$\begin{aligned}\Psi_n(x_1,x_2) &= \left(1-x_1^{-2}\right)\Phi_n(x_1,x_2)\\ &= \left(1-x_1^{-2}\right)\left(x_1+x_1^{-1}+x_2+x_2^{-1}\right)^n \end{aligned}\tag{27}$$

Remark. Making the substitutions $x_1 = e^{i\phi_1}$ and $x_2 = e^{i\phi_2}$ in (25) and (27) allows the introduction of Fourier methods into the study of these generating functions. This leads rapidly into the general theory of random walks (see for example Percus [2]).

§4. Direct Proofs

In this final section we define a bijection between the set $\mathcal{P}_n$ of half-planar n-walks and the set $\mathcal{B}_n$ of linear $(2n+1)$-walks which terminate at $x = 1$, i.e. which have n **R** terms and $n+1$ **L** terms. The cardinality $|\mathcal{B}_n|$ is evidently $\binom{2n+1}{n}$ and hence we will have shown directly that the number of half-planar n-walks is $\binom{2n+1}{n}$.

We give an algorithm which defines a map $f : \mathcal{P}_n \longrightarrow \mathcal{B}_n$. Given an **NSEW** sequence which defines a half-planar n-walk we perform the steps described below. The aim of the first two steps is to replace the original **NSEW** sequence, in which the number of **E**'s may greatly exceed the number of **W**'s, by one in which the number of **E**'s equals, or exceeds by exactly one, the number of **W**'s. The resulting planar walk will thus terminate either on $x = 0$ or $x = 1$. The next two steps convert the planar walk to a linear walk of the required form. The steps of the algorithm are as follows.

(1) **Match.** Working right to left, match each **E** with the closest as yet unmatched **W** to its right, if any, but mark it "unmatched" if none;

(2) **Switch.** (Long method.) Working left to right, for each unmatched **E** term, replace every **E** to its left by **W** and every **W** to its left by **E**. [Thus at each stage, ignoring terms to the right, we have at most one more **E** term than **W** term.]

(Short Cut.) The above is equivalent to working left to right, replacing **E** by **W** or **W** by **E**, whenever the number of unmatched **E** terms to its right is odd;

(3) **Replace.** Replace each **E**, **W**, **N**, **S** by **RR**, **LL**, **LR**, **RL** respectively;

(4) **Final Balance.** Precede the sequence by **R** or **L** according as there were an even or odd number of unmatched **E** terms.

This gives an element of $\mathcal{B}_n$.

Example 8. In this example and the next we indicate an unmatched **E** by **E*** and indicate matched pairs by subscripts.

	NSEWEEESWWEE	$\in \mathcal{P}_{12}$
$\longrightarrow$	$\mathbf{NSE_3W_3E^*E_2E_1SW_1W_2E^*E^*}$	(**Match**)

$\longrightarrow$ NSWEEEESWWWE (**Switch** (forget matchings))
$\longrightarrow$ LRRLLLRRRRRRRRRLLLLLLLRR (**Replace**)
$\longrightarrow$ LLRRLLLRRRRRRRRRLLLLLLLRR (**Final Balance**)$\in \mathcal{B}_{12}$

Next we define the inverse of f by the *composition* of the following processes, which clearly invert the steps above:

(1)' **Drop balance term.** Discard the first letter;

(2)' **Replace.** Replace each pair **RR**, **LL**, **LR**, **RL** by **E**, **W**, **N**, **S** respectively;

(3)' **Switch.** Working right to left, match each **E** with the closest as yet unmatched **W** to its right, if any. If there is none, proceeding to the left, replace every **E** by **W**, and **W** by **E**.

(4)' **Drop match.** Ignore the matching.

This gives an element of $\mathcal{P}_n$.

Example 9.

LLRRLLLRRRRRRRRRLLLLLLLRR $\in \mathcal{B}_{12}$
$\longrightarrow$ LRRLLLRRRRRRRRRLLLLLLLRR (**Drop**)
$\longrightarrow$ NSWEEEESWWWE (**Replace**)
$\longrightarrow$ $\mathrm{NSEWWWW_1SE_1EEE^*}$
$\longrightarrow$ $\mathrm{NSWEEE_2E_1SW_1W_2E^*E^*}$ (**Switch**)
$\longrightarrow$ $\mathrm{NSE_3W_3E^*E_2E_1SW_1W_2E^*E^*}$
$\longrightarrow$ NSEWEEESWWEE (**Ignore matching**)$\in \mathcal{P}_{12}$

The Catalan numbers via equivalence relations

Finally, we will give an independent proof that the number of correctly nested sequences of n pairs of brackets is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Let $\mathcal{C}_n$ be the set of correctly nested sequences of n pairs of brackets (i.e. $2n$-term sequences with n left and n right brackets, such that for all $i \leq 2n$ the first i terms contain at least as many left as right parentheses).

Let $\mathcal{L}_n$ be the set of all linear $2n$-walks which return to the origin.

We define a relation $\sim$ on $\mathcal{L}_n$ as follows:

For $b, c \in \mathcal{L}_n$, $b \sim c$ when c can be obtained from b by,

(i) preceding b by the step **L** (forming a linear $(2n+1)$-walk ending at $x = -1$;

(ii) cycling the terms of this $(2n+1)$-term sequence of steps so that one of its **L**'s becomes the first term;

(iii) dropping this first term.

It is easy to show that $\sim$ is an equivalence relation on $\mathcal{L}_n$, and that each equivalence class contains $n+1$ members, one and only one of which has the property that for all $i \leq 2n$ the first i terms contain at least as many **L**'s as **R**'s.

Let $\mathcal{D}_n$ be the set of equivalence classes (i.e. $\mathcal{L}_n$ factored by $\sim$).

So,

$$|\mathcal{C}_n| = |\mathcal{D}_n| = \frac{1}{n+1}|\mathcal{L}_n| = \frac{1}{n+1}\binom{2n}{n}.$$

References

[1] Cohen, Daniel I. A., *Basic Techniques of Combinatorial Theory*, John Wiley & Sons, 1978.

[2] Percus, J. K., *Combinatorial Methods*, Springer-Verlag Applied Mathematical Sciences Series #4, 1971.

[3] Sands, W., *Personal Communication*, June 1990.