Chains, Multichains and Möbius Numbers

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ABSTRACT. For any finite poset P and any integer $k \geq 0$, let $\alpha_k(P)$ denote the number of k-chains (i.e. chains of cardinality k or length $k-1$) in P. The polynomial $\alpha(P,X) = \sum_{k\geq 0} \alpha_k(P)X^k$ will be referred to as the *chain generating polynomial* of P. Our first results determine the chain generating polynomial $\alpha(P \otimes Q, X)$ and the multichain generating series $m(P \otimes Q, X)$ of the ordinal product $P \otimes Q$, for any two finite posets P and Q. Using this we determine the Möbius function for certain ordinal products. For any integer $n \geq 1$ let $[n]$, B_n, D_n, $L_n(q)$ and Π_n denote the lattices defined on page 97 of [2]. When P is one of the posets $[n]$, B_n or D_n the values of $\alpha_k(P)$ for any $k \geq 0$ are well-known and can easily be determined. When $P = L_n(q)$ or Π_n we will establish recurrence relations in this paper which will effectively determine $\alpha_k(P)$. For example, we explicitly determine $\alpha_k(L_n(q))$ for all $k \geq 0$ when $n \leq 3$. We also obtain recurrence relations for the number $m_k(P)$ of k-multichains of P when $P = L_n(q)$ or Π_n. Using these we explicitly determine $m_k(L_n(q))$ for $n \leq 4$.

§1. Introduction

Throughout this paper all the posets considered are finite. For any integer $k \geq 1$, by a k-chain (resp. a k-multichain) in P we mean a k-tuple $(x_1, x_2, \dots x_k)$ of elements from P satisfying $x_1 < x_2 < \cdots < x_k$ (resp. $x_1 \leq x_2 \leq \cdots \leq x_k$). By definition, the empty set is the only 0-chain or 0-multichain in P. For any integer $k \geq 0$ let $\alpha_k(P)$ (resp. $m_k(P)$) denote the number of k-chains (resp. k-multichains) in P. Thus $\alpha_0(P) = m_0(P) = 1$. For any set A we denote the cardinality of A by $|A|$.

Clearly $\alpha_k(P) = 0$ for $k > |P|$. (However, if $|P| \geq 1$, then $m_k(P) \geq 1$ for all $k \geq 0$.)

Definition 1.1 The polynomial,

$$\alpha(P,X) = \sum_{k\geq 0} \alpha_k(P)X^k = 1 + \sum_{k\geq 1} \alpha_k(P)X^k,$$

will be called the *chain generating polynomial* of P.

The following is well-known.

Proposition 1.2 For any poset P, we have,

(i) $m_0(P) = \alpha_0(P) = 1$;

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(ii) $m_k(P) = \sum_{j=1}^{k} \binom{k-1}{j-1} \alpha_j(P)$ for all $k \geq 1$;

(iii) $\alpha_k(P) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} m_j(P)$ for all $k \geq 1$.

Proof. Part (i) is from the definition. For (ii) refer to Proposition 3.11.1 (a) in [2]. Also it is well-known that (ii) and (iii) are equivalent. ∎

The Möbius function μ_P of a poset is defined as follows:

$$\mu_P(x,y) = \begin{cases} 1 & \text{for } x = y \in P; \\ -\sum_{x \leq z < y} \mu_P(x,z) & \text{for } x < y; \\ 0 & \text{if } x \not\leq y. \end{cases} \tag{1.3}$$

For any poset P we denote the poset obtained by adjoining $\hat{0}$ and $\hat{1}$ to P by $\hat{P}$. Thus $\hat{P} = P \dot{\cup} \{\hat{0}, \hat{1}\}$; a disjoint union with the obvious partial order. The *Möbius number* of P, denoted by $\mu(P)$, is defined by

$$\mu(P) = \sum_{k \geq 0} (-1)^{k-1} \alpha_k(P). \tag{1.4}$$

By a result of G-C. Rota [1] we know that,

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \mu(P). \tag{1.5}$$

The order complex $\Delta(P)$ is the simplicial complex whose k simplices are the $(k+1)$-chains of P. Then the reduced Euler characteristic $\tilde{\chi}(\Delta(P))$ of $\Delta(P)$ is the same as $\sum_{k \geq 0} (-1)^{k-1} \alpha_k(P)$. Hence Rota's result establishes a link between Combinatorics and Algebraic Topology. Many deep results have been obtained in Combinatorics and Commutative Algebra using this link. Also (1.5) has been of immense use in computing the Möbius functions of many posets. For a nice account the reader may refer to §3.8-10 of [2]. However, if one is interested in calculating $\alpha_k(P)$ for a specific poset P, calculations involving the Euler characteristic will not be of much use. The purpose of this paper is to develop combinatorial techniques for the calculation of $\alpha_k(P)$ and $m_k(P)$ and to apply these in several specific cases.

§2. Chain Generating Polynomials and Möbius Numbers of Certain Posets

Let P, Q be two given posets. Let $P + Q$, $P \oplus Q$, $P \times Q$, and $P \otimes Q$ denote respectively the sum, ordinal sum, direct product and the ordinal product of P and Q as defined in §3.2 of [2]. The following are well-known and easy to prove:

- ⋆ $\alpha_k(P+Q) = \alpha_k(P) + \alpha_k(Q)$ (resp. $m_k(P+Q) = m_k(P) + m_k(Q)$), for any $k \geq 1$;
- ⋆ $\alpha_k(P \oplus Q) = \sum_{r=0}^{k} \alpha_r(P)\alpha_{k-r}(Q)$ (resp. $m_k(P \oplus Q) = \sum_{r=0}^{k} m_r(P) m_{k-r}(Q)$), for any $k \geq 0$.

These yield:

$$\alpha(P+Q,X)=\alpha(P,X)+\alpha(Q,X)-1 \tag{2.1}$$
$$\alpha(P\oplus Q,X)=\alpha(P,X)\alpha(Q,X) \tag{2.2}$$

Observing that $\mu(P)=-\alpha(P,-1)$ for any poset P, these in turn yield:

$$\mu(P+Q)=\mu(P)+\mu(Q)+1 \tag{2.3}$$
$$\mu(P\oplus Q)=-\mu(P)\mu(Q) \tag{2.4}$$

$$\mu_{P+Q}(x,y)=\begin{cases}\mu_P(x,y) & \text{if } x,\ y \text{ are in } P\\ \mu_Q(x,y) & \text{if } x,\ y \text{ are in } Q\\ 0 & \text{otherwise.}\end{cases} \tag{2.5}$$

$$\mu_{P\oplus Q}(x,y)=\begin{cases}\mu_P(x,y) & \text{if } x,\ y \text{ are in } P\\ \mu_Q(x,y) & \text{if } x,\ y \text{ are in } Q\\ \mu(P_{>x}\oplus Q_{<y})=-\mu(P_{>x})\mu(Q_{<y}) & \text{if } x\in P \text{ and } y\in Q.\end{cases} \tag{2.6}$$

Here $P_{>x}=\{x'\in P|x'>x\}$ and $Q_{<y}=\{y'\in Q|y'<y\}$. Moreover, by (1.5), $\mu(P_{>x})=\mu_{\hat{P}}(x,\hat{1}_{\hat{P}})$ and $\mu(Q_{<y})=\mu_{\hat{Q}}(\hat{0}_{\hat{Q}},y)$. Actually, (2.6) is an immediate consequence of (2.4).

$$\mu_{P\times Q}\big((a,b),(c,d)\big)=\mu_P(a,c)\mu_Q(b,d). \tag{2.7}$$

A proof of (2.7) is given on page 118 of [2].

$$m_k(P\times Q)=m_k(P)m_k(Q),\quad \text{for all } k\geq 0. \tag{2.8}$$

Lemma 2.9 (i) $\alpha_0(P\times Q)=1$ and

(ii) $$\alpha_k(P\times Q)=\sum_{j=1}^{k}\sum_{h=1}^{j}\sum_{i=1}^{j}(-1)^{k-j}\binom{k-1}{j-1}\binom{j-1}{h-1}\binom{j-1}{i-1}\alpha_h(P)\alpha_i(Q) \tag{2.10}$$

Proof. Part (i) is obvious. Part (ii) follows from Proposition 1.2 (ii) and (iii). ∎

Theorem 2.11 For any two posets P, Q we have

$$\alpha(P\otimes Q,X)=\alpha\big(P,\alpha(Q,X)-1\big).$$

Proof. Let k be an integer ≥ 1. Given any r with $1\leq r\leq k$, any r-composition $(i_1,\ldots,i_r)$ of k, any r-chain $x_1<x_2<\cdots<x_r$ in P and any i_j-chains $a_1^j<\cdots<a_{i_j}^j$ in Q, for $1\leq j\leq r$, then it is clear that

$$(x_1,a_1^1)<\cdots<(x_1,a_{i_1}^1)<(x_2,a_1^2)<\cdots<(x_2,a_{i_2}^2)<\cdots<(x_r,a_1^r)<\cdots<(x_r,a_{i_r}^r),$$

is a k-chain in $P\otimes Q$. Also, every k-chain in $P\otimes Q$ is uniquely expressible in the above form. Hence

$$\alpha_k(P\otimes Q)=\sum_{r=1}^{k}\sum_{(i_1,\ldots,i_r)\in\Gamma_{r,k}}\alpha_r(P)\alpha_{i_1}(Q)\cdots\alpha_{i_r}(Q), \tag{2.12}$$

where $\Gamma_{r,k}$ denotes the set of r-compositions of k. Let $f(X) = 1 + \sum_{k\geq 1} a_k X^k$ and $g(X) = 1 + \sum_{k\geq 1} b_k X^k$ be any two polynomials (with coefficients from any commutative ring). Then $f(g(X)-1) = 1 + \sum_{k\geq 1} d_k X^k$, where d_k is the coefficient of X^k in

$$a_1\Big(\sum_{s\geq 1} b_s X^s\Big) + a_2\Big(\sum_{s\geq 1} b_s X^s\Big)^2 + \cdots + a_k\Big(\sum_{s\geq 1} b_s X^s\Big)^k.$$

Writing $\big(\sum_{s\geq 1} b_s X^s\big)^r$ as $\big(\sum_{s\geq 1} b_s X^s\big)\big(\sum_{s\geq 1} b_s X^s\big)\cdots\big(\sum_{s\geq 1} b_s X^s\big)$, r times, we see that the coefficient of X^k in $\big(\sum_{s\geq 1} b_s X^s\big)^r$ is the same as $\sum_{(i_1,\ldots,i_r)\in\Gamma_{r,k}} b_{i_1} b_{i_2}\cdots b_{i_r}$. Thus

$$d_k = \sum_{r=1}^{k} a_r \sum_{(i_1,\ldots,i_r)\in\Gamma_{r,k}} b_{i_1} b_{i_2}\cdots b_{i_r}. \tag{2.13}$$

From (2.12) and (2.13) we see that the coefficient of X^k in $\alpha\big(P, \alpha(Q,X)-1\big)$ is precisely $\alpha_k(P\otimes Q)$. Hence

$$\alpha(P\otimes Q, X) = \alpha\big(P, \alpha(Q,X)-1\big).$$

■

Corollary 2.14 The Möbius number of $P\otimes Q$ is given by

$$\mu(P\otimes Q) = -\alpha\big(P, \alpha(Q,-1)-1\big).$$

■

For any poset P let $m(P,X)$ denote the formal power series $\sum_{k\geq 0} m_k(P)X^k$. Observe that if $|P|\geq 1$, $m_k(P)\geq 1$ for all k; hence $m(P,X)$ can never be a polynomial if $P\neq\emptyset$. For $P=\emptyset$, $m(P,X)=1$.

Theorem 2.15 For any two posets P, Q we have

$$m(P\otimes Q, X) = \alpha\big(P, m(Q,X)-1\big).$$

Proof. The proof is similar to the proof of Theorem 2.11. Let k be an integer ≥ 1. Given any r with $1\leq r\leq k$, any r-composition $(i_1,\ldots,i_r)$ of k, any r-chain $x_1 < x_2 < \cdots < x_r$ in P and any i_j-multichains $a_1^j \leq \cdots \leq a_{i_j}^j$ in Q, it is clear that

$$(x_1, a_1^1) \leq \cdots \leq (x_1, a_{i_1}^1) < (x_2, a_1^2) \leq \cdots \leq (x_2, a_{i_2}^2) < \cdots < (x_r, a_1^r) \leq \cdots \leq (x_r, a_{i_r}^r),$$

is a k-multichain in $P\otimes Q$. Also, any k-multichain in $P\otimes Q$ is uniquely expressible in the above form. Hence

$$m_k(P\otimes Q) = \sum_{r=1}^{k} \sum_{(i_1,\ldots,i_r)\in\Gamma_{r,k}} \alpha_r(P) m_{i_1}(Q)\cdots m_{i_r}(Q). \tag{2.16}$$

If $f(X) = 1 + \sum_{k\geq 1} a_k X^k$ and $g(X) = 1 + \sum_{k\geq 1} b_k X^k$ are *formal power series* (with coefficients from any commutative ring) having 1 as the constant term, we have $f(g(X)-1) = 1 + \sum_{k\geq 1} d_k X^k$, where d_k is the coefficient of X^k in

$$a_1\Big(\sum_{s\geq 1} b_s X^s\Big) + a_2\Big(\sum_{s\geq 1} b_s X^s\Big)^2 + \cdots + a_k\Big(\sum_{s\geq 1} b_s X^s\Big)^k.$$

The rest of the proof is the same as for Theorem 2.11, with $m(P \otimes Q, X)$ and $m(Q, X)$ replacing $\alpha(P \otimes Q, X)$ and $\alpha(Q, X)$ respectively and $\alpha(P, X)$ remaining unchanged. ∎

Remark 2.17 From (2.3) and (2.4) we see that the Möbius numbers of $P + Q$ and $P \oplus Q$ depend only on $\mu(P)$ and $\mu(Q)$. Also using the fact that the "unreduced Euler characteristic" $\chi(\Delta(P \times Q))$ is the same as $\chi(\Delta(P))\chi(\Delta(Q))$ we see that $\mu(P \times Q) + 1 = (\mu(P)+1)(\mu(Q)+1)$ from which we get $\mu(P \times Q) = \mu(P)\mu(Q) + \mu(P) + \mu(Q)$. However, later we will see from specific examples that $\mu(P \otimes Q)$ is not just a function of $\mu(P)$ and $\mu(Q)$ alone.

Proposition 2.18 The Möbius function $\mu_{P \otimes Q}$ of $P \otimes Q$ is given by

$$\begin{aligned} &\mu_{P\otimes Q}\big((x,a),(x,a)\big) = 1 \quad \text{for all } (x,a) \in P \otimes Q \\ &\mu_{P\otimes Q}\big((x,a),(x,b)\big) = \mu_Q(a,b) \quad \text{for any } x \in P; a, b \in Q \\ &\mu_{P\otimes Q}\big((x,a),(y,b)\big) = \begin{cases} \mu(Q_{>a})\mu(I \otimes Q)\mu(Q_{<b}) & \text{whenever } x < y \text{ in } P \\ 0 & \text{if } x \not\leq y, \end{cases} \end{aligned} \tag{2.19}$$

where $I = \{z \in P | x < z < y\}$, $Q_{>a} = \{c \in Q | c > a\}$ and $Q_{<b} = \{c \in Q | c < b\}$.

Proof. For any $x \in P$, the map $a \longmapsto (x, a)$ is an order isomorphism of Q onto the subposet $\{x\} \times Q$ of $P \times Q$. Hence we have only to verify the third equation in (2.19). Let $E = \{(u, c) \in P \times Q | (x, a) < (u, c) < (y, b)\}$. Clearly, $E = \{x\} \times Q_{>a} \dot\cup (I \otimes Q) \dot\cup \{y\} \times Q_{<b}$ with the union being disjoint. Moreover any (x, c) in $\{x\} \times Q_{>a}$ satisfies $(x, c) < (u, d)$ for all $(u, d) \in I \otimes Q$ and $(u, d) < (y, c)$ for any $(y, c) \in \{y\} \times Q_{<b}$. Thus $E = (\{x\} \times Q_{>a}) \oplus (I \otimes Q) \oplus (\{y\} \times Q_{<b})$. Hence

$$\mu_{P\otimes Q}\big((x,a),(y,b)\big) = \mu(E) = \mu(Q_{>a})\mu(I \otimes Q)\mu(Q_{<b})$$

from iterated application of (2.4). ∎

J.A. Walker [3] has obtained a formula for $\mu(P +_R Q)$, where R is an order ideal in $P \times Q$. When R is of the form $A \times B$ with A an order ideal in P and B an order ideal in Q, it is easy to obtain the chain generating polynomial of $P +_R Q$. Recall that $P +_R Q$ as a set is the disjoint union of P and Q. The partial order on $P +_R Q$ is given by $x \leq y$ if one of the following is true:

(i) $x \leq y$ in P;

(ii) $x \geq y$ in Q; or

(iii) $(x, y) \in R$.

When $R = A \times B$, any k-chain, for $k \geq 1$, in $P +_R Q$ is either a k-chain in P or a k-chain in Q^* (the dual of Q), or is of the form $x_1 < \cdots < x_i < y_{i+1} > y_{i+2} > \cdots > y_k$, with $x_1 < \cdots x_i$ an i-chain in P, and $y_{i+1} > \cdots > y_k$ in Q and $(x_i, y_{i+1}) \in A \times B$. Since A and B are order ideals in P and Q respectively, we get $x_r \in A$, $y_s \in B$ for $1 \leq r \leq i$, $i + 1 \leq s \leq k$. Hence

$$\alpha_k(P +_R Q) = \alpha_k(P) + \alpha_k(Q) + \sum_{i=1}^{k-1} \alpha_i(A)\alpha_{k-i}(B). \tag{2.20}$$

From (2.20) we immediately get the following:

Proposition 2.21 If $R = A \times B$, where A, B are order ideals in P, Q, respectively, then

$$\begin{aligned}\alpha(P +_R Q, X) &= \alpha(P,X) + \alpha(Q,X) - 1 + \{\alpha(A,X) - 1\}\{\alpha(B,X) - 1\} \\ &= \alpha(P+Q,X) - \alpha(A+B,X) + \alpha(A \oplus B, X).\end{aligned}$$

In particular,

$$\mu(P +_R Q) = \mu(P) + \mu(Q) - \mu(A) - \mu(B) - \mu(A)\mu(B). \tag{2.22}$$

■

Note. Since $\mu(Q) = \mu(Q^*)$ and $P + Q = P +_{\emptyset} Q^*$, $P \oplus Q = P +_{P\times Q^*} Q^*$, Equation (2.22) generalizes (2.3) and (2.4).

Examples 2.23

In the following examples the poset [1] will be regarded both as a chain and an antichain. Also m, n will denote integers ≥ 1.

(i) A non-empty poset P is an antichain if and only if $\alpha(P,X) = 1 + nX$ for some integer $n \geq 1$. In this case $|P| = n$.

(ii) In the poset $[n]$ the k-chains are exactly the k-subsets of $[n]$ (the k-multichains are exactly the k-multisubsets of $[n]$). Thus $\alpha_k([n]) = \binom{n}{k}$ and $m_k([n]) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

(iii) Since $B_n \simeq [2]^n$ (the direct product of n copies of the poset [2]) we see that $m_k(B_n) = (k+1)^n$. In the notation of [2], for any poset P, $m_k(P)$ is the same as $Z(P, k+1)$ (page 129 of [2]). Hence we have a verification of Example 3.11.2 of [2], that $Z(B_n, k+1) = (k+1)^n$. From Proposition 1.2 we get

$$\alpha_0(B_n) = 1 \qquad \text{and} \qquad \alpha_k(B_n) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} (j+1)^n. \tag{2.24}$$

(iv) Let n be an integer ≥ 2 and $n = p_1^{k_1} \cdots p_r^{k_r}$, with $p_1, \ldots, p_r$ distinct primes and $k_1, \ldots, k_r$ integers ≥ 1, for $1 \leq i \leq r$. The map $\phi : D_n \longrightarrow [k_1 + 1] \times \cdots \times [k_r + 1]$ defined by $\phi(p_1^{a_1} \cdots p_r^{a_r}) = (a_1 + 1, \ldots, a_r + 1)$ for $0 \leq a_i \leq k_i$ is known to be an isomorphism of posets. Hence,

$$m_j(D_n) = \binom{k_1 + j}{j}\binom{k_2 + j}{j} \cdots \binom{k_r + j}{j}, \tag{2.25}$$

for all $j \geq 0$ (from (2.23) (ii) and (2.8)).

Proposition 2.26 Let n be an integer ≥ 1. For any integer k satisfying $1 \leq k \leq n+1$ let G_k be the set of k-tuples $(i_1, i_2, \ldots, i_k)$ of integers satisfying the conditions:

$$\begin{aligned}&0 \leq i_1 \leq n - (k-1) \\ &1 \leq i_1 + i_2 \leq n - (k-2) \\ &1 \leq i_1 + i_2 + i_3 \leq n - (k-3) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &1 \leq i_1 + i_2 + \cdots + i_k \leq n\end{aligned} \tag{2.27}$$

Then we have

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j-1}(j+1)^n = \sum_{(i_1,\ldots,i_k)\in G_k}\binom{n}{i_1,i_2,\ldots,i_k,n-\sum_{j=1}^{k}i_j}, \tag{2.28}$$

for any k satisfying $1 \le k \le n+1$, where the terms on the right-hand side of (2.28) are multinomial coefficients.

If $k \ge n+2$ we have

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j-1}(j+1)^n = 0. \tag{2.29}$$

Proof. The proof consists in calculating the number of k-chains $\alpha_k(B_n)$ in two different ways. Observing that $B_n = J(n[1])$ where $n[1]$ is the antichain with n elements, we see that B_n is a graded poset of rank n. Hence $\alpha_k(B_n) = 0$ for $k \ge n+2$. This combined with (2.24) immediately yields (2.29). Denoting the elements of the poset $n[1]$ by $a_1,\ldots,a_n$, for $1 \le k \le n+1$, a typical k-chain in B_n will be of the form $E_1 \subset E_2 \subset \cdots \subset E_k$ with $E_j \subset \{a_1,\ldots,a_n\}$, $|E_1| \ge 0$, $|E_{j+1}| > |E_j|$ for $1 \le j \le k-1$. The number of such k-chains with $|E_1| = i_1$, $|E_{j+1}|-|E_j| = i_{j+1}$ for $1 \le j \le k-1$ is precisely $\binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-\sum_{j=1}^{k-1}i_j}{i_k} = \binom{n}{i_1,i_2,\ldots,i_k,n-\sum_{j=1}^{k}i_j}$. Since each E_j is a subset of $\{a_1,a_2,\ldots,a_n\}$ and $i_j \ge 1$ for $2 \le j \le k$ we see immediately that $i_1,\ldots,i_k$ have to satisfy the restrictions of (2.27). It follows that

$$\alpha_k(B_n) = \sum_{(i_1,\ldots,i_k)\in G_k}\binom{n}{i_1,i_2,\ldots,i_k,n-\sum_{j=1}^{k}i_j}$$

for $1 \le k \le n+1$. Combining this with (2.24) we immediately get (2.28). ∎

Examples 2.30

(i) Let A_m denote the antichain with m elements. For any poset P, it is easy to see that $A_m \otimes P \simeq P + \cdots + P$ (m-copies). From repeated application of (2.1) we get

$$\begin{aligned}\alpha(A_m \otimes P, X) &= m\alpha(P,X) - (m-1)\\ &= 1 + m\big(\alpha(P,X) - 1\big),\end{aligned} \tag{2.31}$$

which agrees with the calculation of $\alpha(A_m \otimes P, X)$ obtained from Theorem 2.11. Theorem 2.11 also yields $\alpha(A_m \otimes A_n, X) = 1 + mnX$, which confirms the fact that $A_m \otimes A_n \simeq A_{mn}$. Finally, from Theorem 2.11 we immediately get

$$\alpha\big([m]\otimes A_n, X\big) = (1+nX)^m \tag{2.32}$$

$$\alpha\big([m]\otimes [n], X\big) = (1+X)^{mn} \tag{2.33}$$

Clearly $[m]\otimes[n] \simeq [mn]$ and formula (2.33) is in agreement with this fact. From (2.32) we see that

$$\mu\big([m]\otimes A_n\big) = -(1-n)^m. \tag{2.34}$$

For any $m \ge 1$, we know that $\mu([m]) = 0$ and for any $n \ge 1$, $\mu(A_n) = n-1$. Formula (2.34) shows that $\mu\big([m]\otimes A_n\big)$ depends also on m and not just on $(n-1)$ alone. This illustrates the last part of our earlier Remark 2.17.

(ii) As an exercise we explicitly evaluate the Möbius function of the poset $[m] \otimes A_n$ using Proposition 2.18.

$$\mu_{[m]\otimes A_n}((i,a),(j,b)) = \begin{cases} 1 & \text{if } i=j \text{ and } a=b; \\ 0 & \text{if } i=j \text{ and } a \neq b; \\ -1 & \text{if } j=i+1; \\ -(1-n)^{j-i-1} & \text{if } i+1<j \text{ in } [m]; \\ 0 & \text{if } i \not\leq j \text{ in } [m]. \end{cases}$$

§3. Recurrence Relations Satisfied by $\alpha_k(L_n(q))$ and $m_k(L_n(q))$

In this section q will denote a power of a prime and F_q the finite field with q elements. n will denote an integer ≥ 0 and $V_n(q)$ the vector space of dimension n over F_q. We write $L_n = L_n(q)$ for the lattice of subspaces of $V_n(q)$. For any integer j satisfying $0 \leq j \leq n$, $\begin{bmatrix} n \\ j \end{bmatrix}$ will denote the q-binomial coefficient; namely

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \qquad \text{for all} \qquad n \geq 0,$$
$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{j-1})}{(q^j-1)(q^j-q)\cdots(q^j-q^{j-1})} \quad \text{if } 1 \leq j \leq n.$$

For $j > n$, by convention $\begin{bmatrix} n \\ j \end{bmatrix} = 0$. As is well-known [c.f. Proposition 1.3.18 of [2]], $\begin{bmatrix} n \\ j \end{bmatrix}$ is equal to the number of j-dimensional subspaces of $V_n(q)$. For any finite-dimensional vector space V we write $L(V)$ for the lattice of vector subspaces of V.

For any integer $k \geq 0$, let $\alpha_{k,n}$ denote the number of k-chains in L_n. For $k \geq 1$, let $\gamma_{k,n}$ be the number of k-chains $W_1 \subset W_2 \subset \cdots \subset W_k$ in L_n with $W_1 = 0$.

By convention $\alpha_{0,n} = 1$. (3.1)

Also it is clear that

$$\gamma_{1,n} = 1 \qquad \text{for all } n \geq 0 \quad \text{and } \alpha_{k,n} = \gamma_{k,n} = 0 \; if \; k > n+1. \tag{3.2}$$

Moreover, $\alpha_{1,n}$ is the total number of vector subspaces of $V_n(q)$. Hence,

$$\alpha_{1,n} = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} + \begin{bmatrix} n \\ n \end{bmatrix}. \tag{3.3}$$

Proposition 3.4 For any $k \geq 1$, we have

$$\alpha_{k,n} = \gamma_{k,n} + \begin{bmatrix} n \\ 1 \end{bmatrix}\gamma_{k,n-1} + \cdots + \begin{bmatrix} n \\ n-k \end{bmatrix}\gamma_{k,k} + \begin{bmatrix} n \\ n-k+1 \end{bmatrix}\gamma_{k,k-1}. \tag{3.5}$$

Proof. Given a vector subspace W of $V_n = V_n(q)$, there is a bijective correspondence between k-chains $W_1 \subset W_2 \subset \cdots \subset W_k$ in L_n with $W_1 = W$ and k-chains $\bar{W}_1 \subset \bar{W}_2 \subset \cdots \subset \bar{W}_k$ in $L(V_n/W)$ satisfying $\bar{W}_1 = 0$. If $\dim W = j$ then $\dim V_n/W = n-j$. Hence the number of k-chains $W_1 \subset W_2 \subset \cdots \subset W_k$ in L_n with $\dim W_1 = j$ is the same as $\begin{bmatrix} n \\ j \end{bmatrix}\gamma_{k,n-j}$. Summing up we get (3.5). ∎

Lemma 3.6 For any integer $k \geq 2$, we have

$$\gamma_{k,n} = \alpha_{k-1,n} - \gamma_{k-1,n}. \tag{3.7}$$

Proof. Any k-chain $W_1 \subset W_2 \subset \cdots \subset W_k$ in L_n with $W_1 = 0$ yields a $(k-1)$-chain $\bar{W}_2 \subset \bar{W}_3 \subset \cdots \subset \bar{W}_k$ in L_n with $W_2 \neq 0$. Conversely, given any $(k-1)$-chain $W_2 \subset W_3 \subset \cdots \subset W_k$ with $W_2 \neq 0$ in L_n, we see that $0 \subset W_2 \subset W_3 \subset \cdots \subset W_k$ is a k-chain in L_n with the first element the zero subspace of V_n. Clearly these processes are inverses of one another. The number of $(k-1)$-chains $W_2 \subset W_3 \subset \cdots \subset W_k$ in L_n with $W_2 \neq 0$ is clearly $\alpha_{k-1,n} - \gamma_{k-1,n}$. Hence the result. ■

Remark 3.8 Equations (3.2) and (3.3) give us the values of $\gamma_{1,n}$ and $\alpha_{1,n}$ for all n. Also (3.7) gives $\gamma_{2,n}$ for all n and then (3.5) will give us $\alpha_{2,n}$ for all n. Assuming that for a certain $k \geq 3$ we have determined $\gamma_{k-1,n}$ and $\alpha_{k-1,n}$ using (3.7) we can determine $\alpha_{k,n}$ for all n. Thus the recurrence relations (3.5) and (3.7) together with the initial conditions (3.1), (3.2) and (3.3) effectively determine $\alpha_{k,n}$ for all k and all n. For instance,

$$\gamma_{2,n} = \alpha_{1,n} - \gamma_{1,n} = 1 + \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix}$$

and,

$$\alpha_{2,n} = \gamma_{2,n} + \begin{bmatrix} n \\ 1 \end{bmatrix} \gamma_{2,n-1} + \cdots + \begin{bmatrix} n \\ n-2 \end{bmatrix} \left(1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} n \\ n-1 \end{bmatrix}.$$

In the following table we record the values of $\gamma_{k,n}$ and $\alpha_{k,n}$ for $n \leq 3$. These are calculated following the procedure described above.

k	$\alpha_{k,0}$	$\alpha_{k,1}$	$\alpha_{k,2}$	$\alpha_{k,3}$	$\gamma_{k,0}$	$\gamma_{k,1}$	$\gamma_{k,2}$	$\gamma_{k,3}$
1	1	2	$q+3$	$2q^2+2q+4$	1	1	1	1
2	0	1	$2q+3$	q^3+6q^2+6q+5	0	1	$q+2$	$2q^2+2q+3$
3	0	0	$q+1$	$2q^3+6q^2+6q+3$	0	0	$q+1$	q^3+4q^2+4q+2
4	0	0	0	q^3+2q^2+2q+1	0	0	0	q^3+2q^2+2q+1
5	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

$\alpha_{k,n} = \alpha_k(L_n)$ and $\gamma_{k,n}$ for small values of k and n.

From Proposition 1.2 we know that for any poset P we can find all the $m_k(P)$'s if we know all the $\alpha_k(P)$'s and vice versa. However if $P = L_n$, we are able to obtain recurrence relations between the $m_k(L_n)$'s, simplifying their calculation.

Let $m_{k,n} = m_k(L_n)$. By definition we have $m_{0,n} = 1$ for all $n \geq 0$ and it is easy to see that

$$m_{k,0} = 1 \qquad \text{for all } k \geq 0. \tag{3.9}$$

Proposition 3.10 For any $k \geq 1$,

$$m_{k,n} = m_{k-1,n} + \begin{bmatrix} n \\ 1 \end{bmatrix} m_{k-1,n-1} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} m_{k-1,1} + m_{k-1,0}. \tag{3.11}$$

Proof. Given a vector subspace W of V_n, there is a bijective correspondence between k-multichains $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_k$ in L_n with $W_1 = W$ and $(k-1)$-multichains $\bar{W}_1 \subseteq \bar{W}_2 \subseteq \cdots \subseteq \bar{W}_k$ in $L(V_n/W)$. Hence the number of k-multichains $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_k$ in L_n with $\dim W_1 = j$ is the same as $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right] m_{k-1,n-j}$. Summing up we get (3.11). ∎

Since $m_{1,n}$ is the same as the total number of vector subspaces of V_n we have

$$m_{1,n} = 1 + \begin{bmatrix} n \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} + 1. \tag{3.12}$$

Remark 3.13 Using (3.12) and repeatedly applying (3.11) we can calculate the value of $m_{k,n}$ for any k and n.

Proposition 3.14

$$m_{k,n} = 1 + \begin{bmatrix} n \\ 1 \end{bmatrix}\Big(\sum_{j=0}^{k-1} m_{j,n-1}\Big) + \begin{bmatrix} n \\ 2 \end{bmatrix}\Big(\sum_{j=0}^{k-1} m_{j,n-2}\Big) + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix}\Big(\sum_{j=0}^{k-1} m_{j,1}\Big) + k. \tag{3.15}$$

Proof. From Proposition 3.10, using the fact that $m_{k,0} = 1$ for all $k \geq 0$ we get the following equations:

$$\begin{aligned} m_{k,n} &= m_{k-1,n} + \begin{bmatrix} n \\ 1 \end{bmatrix} m_{k-1,n-1} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} m_{k-1,1} + 1 \\ m_{k-1,n} &= m_{k-2,n} + \begin{bmatrix} n \\ 1 \end{bmatrix} m_{k-2,n-1} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} m_{k-2,1} + 1 \\ \cdots &= \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ m_{1,n} &= 1 + \begin{bmatrix} n \\ 1 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} + 1 \end{aligned}$$

Adding all these equations and cancelling $\sum_{j=1}^{k-1} m_{j,n}$ on both sides we get equation (3.15). ∎

Remark 3.16

(i) Any k-multichain in L_1 is of the form $0 \subseteq \cdots \subseteq 0 \subseteq V_1 \subseteq \cdots \subseteq V_1$, with i zero subspaces and $(k-i)$ subspaces equal to the full vector space V_1, for $0 \leq i \leq k$. Hence,

$$m_{k,1} = k + 1. \tag{3.17}$$

(ii) Using (3.17), (3.9) and (3.15) we get

$$m_{k,2} = (k+1) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\binom{k+1}{2} = (k+1) + \frac{(q+1)k(k+1)}{2}. \tag{3.18}$$

(iii) Now,

$$\sum_{j=0}^{k-1} m_{j,1} = \binom{k+1}{2}$$

and, $$\sum_{j=0}^{k-1} m_{j,2} = \sum_{j=0}^{k-1}\left\{(j+1)+\begin{bmatrix}2\\1\end{bmatrix}\binom{j+1}{2}\right\} = \binom{k+1}{2}+\binom{k+1}{3}\begin{bmatrix}2\\1\end{bmatrix}$$

Hence we have from (3.15),

$$m_{k,3} = (k+1)+\binom{k+1}{2}\left\{\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}\right\}+\binom{k+1}{3}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}. \tag{3.19}$$

(iv) Similarly,

$$\sum_{j=0}^{k-1} m_{j,3} = \binom{k+1}{2}+\binom{k+1}{3}\left(\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}\right)+\binom{k+1}{4}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix},$$

hence, $$\begin{aligned} m_{k,4} = (k+1) &+ \binom{k+1}{2}\left(\begin{bmatrix}4\\1\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}\right) \\ &+ \binom{k+1}{3}\left(\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\right) \\ &+ \binom{k+1}{4}\begin{bmatrix}4\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}. \end{aligned}$$

§4. Recurrence Relations Satisfied by $\alpha_k(\Pi_n)$ and $m_k(\Pi_n)$

In this section n will denote an integer ≥ 1, Π_n will denote the lattice of partitions of the set $[n]$ and k an integer ≥ 0. The element $\hat{0}$ of Π_n is the partition $(\{1\},\{2\},\ldots,\{n\})$ of $[n]$. Let $\beta_{k,n} = \alpha_k(\Pi_n)$ for $k \geq 0$ and for $k \geq 1$ let $\eta_{k,n}$ be the number of k-chains in Π_n with the zero element of Π_n as the first element. By definition $\beta_{0,n} = 1$ for all $n \geq 1$.

Proposition 4.1 For any $k \geq 1$, we have

$$\beta_{k,n} = \eta_{k,n} + S(n,n-1)\eta_{k,n-1} + \cdots + S(n,2)\eta_{k,2} + S(n,1)\eta_{k,1}, \tag{4.2}$$

where $S(n,j)$ are Stirling numbers of the second type.

Proof. Let π be any element of Π_n. If $|\pi|$ denotes the number of elements in the partition, then it is known that the interval $[\pi,\hat{1}]$ in Π_n is isomorphic to Π_r where $r = |\pi|$ (page 128 of [2]). Also Π_n is a graded poset; the rank $\rho(\pi)$ of π is given by $\rho(\pi) = n - |\pi|$. It is clear that the number of k-chains in $[\pi,\hat{1}]$ which begin at the zero element of the poset $[\pi,\hat{1}]$ is the same as the number of k-chains in Π_n beginning at π. The number of elements of rank p in Π_n is $S(n,n-p)$. Thus summing up over elements of rank p, $0 \leq p \leq n-1$ (the rank of Π_n is $n-1$) we get the equation 4.2.

Proposition 4.3 For $k \geq 2$, we have

$$\eta_{k,n} = \beta_{k-1,n} - \eta_{k-1,n}. \tag{4.4}$$

Proof. The proof is similar to that of Lemma 3.6 and hence omitted. ∎

Remark 4.5 It is clear that for every $n \geq 1$,

$$\eta_{1,n} = 1. \tag{4.6}$$

Also, $\beta_{1,n}$ is the number of elements in Π_n, which is the nth Bell number $B(n)$ (page 33 of [2]). (4.7)

If $k \geq 2$ and if we know the values of $\beta_{k-1,n}$ and $\eta_{k-1,n}$ using (4.4) we get the values of $\eta_{k,n}$ and then using (4.2) we get the values of $\beta_{k,n}$.

For $n \leq 3$, we can determine $\beta_{k,n}$ easily. Π_1 is the poset with only one element. Thus $\beta_{1,1} = 1$ and $\beta_{k,1} = 0$ for $k \geq 2$. Π_2 is the poset $\{\hat{0}, \hat{1}\}$ with two elements (namely $(\{1\},\{2\}) = \hat{0} \in \Pi_2$ and $\{1,2\} = \hat{1} \in \Pi_2$). Thus, $\beta_{1,2} = 2$, $\beta_{2,2} = 1$ and $\beta_{k,2} = 0$ for $k \geq 3$.

For the poset Π_3, we easily see that $\beta_{0,3} = 1$, $\beta_{1,3} = 5$, $\beta_{2,3} = 7$, $\beta_{3,3} = 3$, and $\beta_{k,3} = 0$ for $k \geq 4$.

For the number $m_k(\Pi_n)$ of k-multichains in Π_n we have the following type of recurrence relation.

Proposition 4.8 For any integer $k \geq 1$ we have

$$m_k(\Pi_n) = m_{k-1}(\Pi_n) + S(n, n-1)m_{k-1}(\Pi_{n-1}) + \cdots + S(n,1)m_{k-1}(\Pi_1). \tag{4.9}$$

Proof. For any $\pi \in \Pi_n$, the number of k-multichains in Π_n beginning with π is the same as the number of $(k-1)$-multichains in the interval $[\pi, \hat{1}]$. Since $[\pi, \hat{1}] \simeq \Pi_{n-\rho(\pi)}$, where $\rho(\pi)$ is the rank of π, we get equation 4.9 by summing up over elements of rank ρ with $0 \leq \rho \leq n-1$. ∎

Since Π_1 is the poset with just one element we have $m_k(\Pi_1) = 1$ for all k. Also $m_1(\Pi_n) = B(n)$, the nth Bell number.

Proposition 4.10 For any $k \geq 1$, we have

$$m_k(\Pi_n) = 1 + S(n, n-1)\Big(\sum_{j=0}^{k-1} m_j(\Pi_{n-1})\Big) + \cdots + S(n,2)\Big(\sum_{j=0}^{k-1} m_j(\Pi_2)\Big) + k \tag{4.11}$$

Proof. Similar to the proof of Proposition 3.14, hence omitted. ∎

Since Π_2 is the set with $\hat{0}$, $\hat{1}$ as the only two elements, we see that $m_k(\Pi_2) = (k+1)$.

From (4.11) we see that

$$\begin{aligned} m_k(\Pi_3) &= 1 + S(3,2)(1 + 2 + \cdots + k) + k \\ &= 1 + \frac{3k(k+1)}{2} + k. \end{aligned}$$

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