The Languages of K, T, B, S4 and S5

Syntax

No change.

Semantics

We go more general by introducing an *accessibility* relation on the set of worlds W – that is, a set of ordered pairs <w_1, w_2> where w_1 and w_2 are elements of W (intuitively, w_2 is accessible from w_1 – there is an arrow from w_1 to w_2). Then we modify the interpretation of '□' and '◊' as follows:

(1) m('□') = the *f* such that *f*(*p*) is true at $w \in$W iff *p* is true at every $w' \in$W that is accessible from *w*.
(2) m('◊') = the *f* such that *f*(*p*) is true at $w \in$W iff *p* is true at some $w' \in$W that is accessible from *w*.

Before we had: '□φ' is true at a world *w* iff φ is true at every possible world, and '◊φ' is true at a world *w* iff φ is true at some possible world. *Now we have*: '□φ' is true at a world *w* iff φ is true at every possible world that is accessible from *w*, and '◊φ' is true at a world *w* iff φ is true at some possible world that is accessible from *w*.

To interpret the language we now need to specify a set of worlds (marking out one as the actual world) and a truth value for each sentence letter in each world (as before), but now we *also* need to specify *an accessibility relation on those worlds*.

We distinguish *five* kinds of interpretation, and hence *five* languages:

- A **K**-interpretation is any interpretation.
- A **T**-interpretation is one whose accessibility relation is *reflexive*.
- A **B**-interpretation is one whose accessibility relation is *reflexive* and *symmetric*.
- An **S4**-interpretation is one whose accessibility relation is *reflexive* and *transitive*.
- An **S5**-interpretation is one whose accessibility relation is *reflexive*, *symmetric*, and *transitive*.

Entailment. If Γ is a set of wffs (possibly empty) and φ is a wff, then we define:

- '$\Gamma \models_K \varphi$' means that there is no K-interpretation on which each wff in Γ is true and yet φ is false. Say that Γ *K-entails* φ. If Γ is empty, say that φ is *K-valid*.
- '$\Gamma \models_T \varphi$' means that there is no T-interpretation on which each wff in Γ is true and yet φ is false. Say that Γ *T-entails* φ. If Γ is empty, say that φ is *T-valid*.
- '$\Gamma \models_B \varphi$' means that there is no B-interpretation on which each wff in Γ is true and yet φ is false. Say that Γ *B-entails* φ. If Γ is empty, say that φ is *B-valid*.
- '$\Gamma \models_{S4} \varphi$' means that there is no S4-interpretation on which each wff in Γ is true and yet φ is false. Say that Γ *S4-entails* φ. If Γ is empty, say that φ is *S4-valid*.
- '$\Gamma \models_{S5} \varphi$' means that there is no S5-interpretation on which each wff in Γ is true and yet φ is false. Say that Γ *S5-entails* φ. If Γ is empty, say that φ is *S5-valid*.

Results:

If φ is a wff then $\Box\varphi \models \varphi$ iff the accessibility relation is required to be *reflexive*.
If φ is a wff then $\varphi \models \Box\Diamond\varphi$ iff the accessibility relation is required to be *symmetric*.
If φ is a wff then $\Box\varphi \models \Box\Box\varphi$ iff the accessibility relation is required to be *transitive*.
If φ is a wff then $\Diamond\varphi \models \Box\Diamond\varphi$ iff the accessibility relation is required to be *Euclidean* (reflexive and symmetric).

For this reason, we say that:

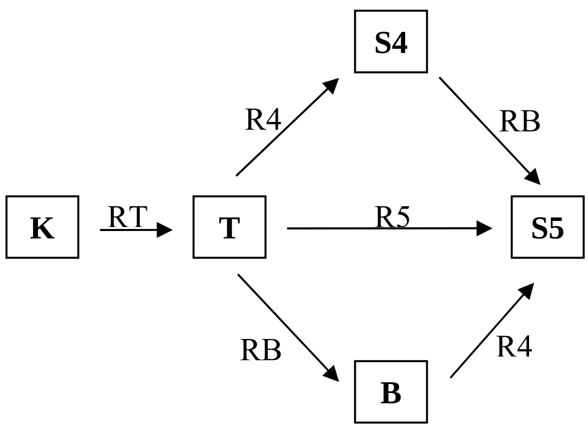
T is characterized by the sequent '$\Box\varphi \models \varphi$', or by the wff '$[\Box\varphi \rightarrow \varphi]$'.
B is characterized by the sequent '$\varphi \models \Box\Diamond\varphi$', or by the wff '$[\varphi \rightarrow \Box\Diamond\varphi]$'.
S4 is characterized by the sequent '$\Box\varphi \models \Box\Box\varphi$', or by the wff '$[\Box\varphi \rightarrow \Box\Box\varphi]$'.
S5 is characterized by the sequent '$\Diamond\varphi \models \Box\Diamond\varphi$', or by the wff '$[\Diamond\varphi \rightarrow \Box\Diamond\varphi]$'.

<u>Deductive systems</u>

K, T, B, S4 and S5 are sound and complete if we add rules of deduction to each as follows:

K	Def $\Diamond$ + N + RK
T	Def $\Diamond$ + N + RK + RT
B	Def $\Diamond$ + N + RK + RT + RB
S4	Def $\Diamond$ + N + RK + RT + R4
S5	Def $\Diamond$ + N + RK + RT + R5, or Def $\Diamond$ + N + RK + RT + RB + R4

The languages are related as follows:



Exercises on K, T, B, S4 and S5

1. Prove the following in K:
   a. $\Box[A \rightarrow B], \Box[B \rightarrow C] \vdash \Box[A \rightarrow C]$
   b. $\Box[A \rightarrow B], \Box A, \Box[B \rightarrow C] \vdash \Box C$
   c. $\Box[A \,\&\, B] \vdash \Box A$
   d. $\Box[A \vee B], \Box\neg A \vdash \Box B$
   e. $\Box[A \vee B], \Box[A \rightarrow B] \vdash \Box B$
   f. $\vdash \Box[A \rightarrow A]$
   g. $\Box[A \rightarrow B], \Box A \vdash \Box B$
   h. $\Box[A \rightarrow B], \Diamond A \vdash \Diamond B$
   i. $\Diamond[A \vee B] \vdash [\Diamond A \vee \Diamond B]$
   j. $\Box\neg A \vdash \neg\Box A$
   k. $\Box\neg A \vdash \Box[A \rightarrow \neg A]$
   l. $\Box\neg A \vdash \Box[A \rightarrow B]]$
   m. $\Box A \vdash [\Diamond B \rightarrow \Diamond[A \,\&\, B]]$
   n. $\Diamond[A \rightarrow B] \vdash [\Box A \rightarrow \Diamond B]$
   o. $\neg\Diamond A \vdash \Box[A \rightarrow B]$
   p. $\Box A \vdash \Box[A \vee B]$
   q. $\vdash [\Box\neg A \leftrightarrow \neg\Diamond A]$
   r. $\vdash [\neg\Diamond A \rightarrow \Box[A \rightarrow B]]$
   s. $\vdash \neg\Diamond[A \,\&\, \neg A]$

2. Prove the following in T:
   a. $\Box A \vdash A$
   b. $A \vdash \Diamond A$
   c. $\vdash [\Box\Diamond A \rightarrow \Diamond A]$
   d. $\vdash [\Box A \rightarrow \Diamond\Box A]$
   e. $\Box[A \rightarrow B], A \vdash B$
   f. $\Box[A \rightarrow B], \Box[A \rightarrow \Box C], \Box[B \rightarrow \Box\neg C] \vdash \Box\neg A$
   g. $\Box[A \vee [B \vee C]], \Box[B \rightarrow D], \Box[[A \vee C] \rightarrow E] \vdash \Box[D \vee C]$
   h. $[\Box[A \rightarrow B] \,\&\, \Box[C \rightarrow B]], \Box[A \vee C] \vdash \Box B$
   i. $\vdash \Box[\Box\Box A \rightarrow \Box A]$
   j. $\vdash \Box[\Box[A \rightarrow B] \rightarrow [\Box A \rightarrow \Box B]]$
   k. $\vdash \Box[[\Box A \,\&\, \Box B] \leftrightarrow \Box[A \leftrightarrow B]]$
   l. $\vdash \Box[\Box A \rightarrow \Box[B \rightarrow A]]$
   m. $\vdash \Box[\Box\neg A \rightarrow \Box[A \rightarrow B]]$
   n. $\vdash \Box[\Box[A \rightarrow [B \,\&\, \neg B]] \rightarrow \Box\neg A]$
   o. $\vdash \Box[[\Box[A \rightarrow B] \,\&\, \Box[\neg A \rightarrow B]] \rightarrow \Box B]$
   p. $\vdash [\Box[A \rightarrow B] \rightarrow [\Diamond A \rightarrow \Diamond B]]$
   q. $\vdash [\Diamond[A \,\&\, B] \rightarrow [\Diamond A \,\&\, \Diamond B]]$
   r. $\vdash [[\Box A \,\&\, \Diamond B] \rightarrow \Diamond[A \,\&\, B]]$
   s. $\vdash [[\Diamond\neg A \vee \Diamond\neg B] \vee \Diamond[A \vee B]]$
   t. $\neg\Diamond A \vdash \neg\Box A$

3. Prove the following in B:
   a. $A \vdash \Box\Diamond A$
   b. $\vdash [\Diamond\Box A \rightarrow A]$

c. $\vdash \Box[A \to \Box\Diamond A]$
d. $\vdash [A \to [\Diamond\Box\Box A \to \Box A]]$

4. Prove the following in S4:
   a. $\Box A \vdash \Box\Box A$
   b. $\vdash [\Box\Box A \leftrightarrow \Box A]$
   c. $\vdash [\Diamond\Diamond A \leftrightarrow \Diamond A]$
   d. $\Diamond\Box\Diamond A \vdash \Diamond A$

5. Prove the following in S5:
   a. $\Diamond A \vdash \Box\Diamond A$
   b. $\vdash [\Diamond A \to \Box\Diamond A]$
   c. $\vdash [\Diamond\Box A \to \Box A]$
   d. $\vdash [\Box\Diamond A \leftrightarrow \Diamond A]$
   e. $\vdash [\Diamond\Box A \leftrightarrow \Box A]$
   f. $\vdash [\Diamond[\Diamond A \,\&\, \neg B] \vee \Box[A \to \Box B]]$