The Language of S5

Syntax

Additional symbols:

Modal operators: ‘□’, ‘◊’.

Additional wffs:

If φ is a wff then so is ‘□φ’ and ‘◊φ’.

Semantics

It is not enough to take the meaning of a wff to be a truth value, because then we cannot give the meaning of ‘□φ’ in terms of the meaning of ‘□’ and the meaning of φ. Instead, we take there to be a set of *possible worlds*, W, and we take the meaning of a wff to be function from possible worlds to truth values (equivalently: a set of possible worlds). Call such a function a *proposition*. So the meaning of a wff is a proposition. The meaning of all wffs are then determined compositionally by the following rules:

(1) The meaning of a sentence letter is a proposition.
(2) m(‘¬’) = the *f* such that *f*(*p*) is true at *w*∈W iff *p* is false at *w*.
(3) m(‘&’) = the *f* such that *f*(*p*, *q*) is true at *w*∈W iff *p* is true at *w* and *q* is true at *w*.
(4) m(‘∨’) = the *f* such that *f*(*p*, *q*) is true at *w*∈W iff *p* is true at *w* or *q* is true at *w*.
(5) m(‘→’) = the *f* such that *f*(*p*, *q*) is true at *w*∈W iff *p* is false at *w* or *q* is true at *w*.
(6) m(‘↔’) = the *f* such that *f*(*p*, *q*) is true at *w*∈W iff *p* is true at *w* and *q* is true at *w*, or *p* is false at *w* and *q* is false at *w*.
(7) m(‘□’) = the *f* such that *f*(*p*) is true at *w*∈W iff *p* is true at all *w*'∈W.
(8) m(‘◊’) = the *f* such that *f*(*p*) is true at *w*∈W iff *p* is true at some *w*'∈W.
(9) If C is a 1-place connective then m(‘Cφ’) = m(C)(m(φ)) (same as for PC)
(10) If C is a 2-place connective then m(‘[φ C ψ]’ = m(C)(m(φ), m(ψ))

Note in particular: ‘□φ’ is true at *w*∈W iff φ is true at *all* *w*'∈W; ‘◊φ’ is true at *w*∈W iff φ is true at *some* *w*'∈W.

The meaning of the 1-place connective (‘¬’) is a 1-place function from propositions to propositions; the meaning of each 2-place connective (‘&’, ‘∨’, ‘→’ , ‘↔’) is a 2-place function from pairs of propositions to propositions.

The connectives are always interpreted in the same way. So interpreting the language amounts to just specifying a set of possible worlds W, and interpreting the sentence letters – that is, assigning a proposition to each.

If we pick out one world in W and call it the actual world, @, then we can define: φ is true (simpliciter) on an interpretation iff φ is true at @ on that interpretation.

Examples of correct semantic sequents: $\Box A \models A$; $A \models \Box\Diamond A$; $\Box A \models \Box\Box A$; $\Diamond A \models \Box\Diamond A$; $A \models \Diamond A$; $\Box[A \to B], \Diamond A \models \Diamond B$; $\models [\Box\neg A \to \neg\Box A]$; $\models [\neg\Diamond A \to \Box[A \to B]]$; $\models \Box[\Diamond\Box A \to \Box\Diamond A]$.

Examples of incorrect semantic sequents: $\Diamond A \models A$; $A \models \Box A$; $[\Diamond A \,\&\, \Diamond B] \models \Diamond[A \,\&\, B]$; $\Diamond A, \Diamond[A \rightarrow B] \models \Diamond B$.

<u>Deductive system</u>

Additional rules of inference:

Def $\Diamond$	From $\Box\varphi$ deduce $\neg\Diamond\neg\varphi$ (and vice-versa) From $\Diamond\varphi$ deduce $\neg\Box\neg\varphi$ (and vv)
RN	From $^{\Box}[\therefore \varphi]$ deduce $\Box\varphi$ (φ not in the scope of an assumption in $^{\Box}[\ldots]$)
RK	From $\Box\varphi$ and $^{\Box}[\ldots]$ deduce $^{\Box}[\ldots\varphi]$ ($\Box\varphi$ not in the scope of an assumption)
RT	From $\Box\varphi$ deduce φ
RB	From φ and $^{\Box}[\ldots]$ deduce $^{\Box}[\ldots\Diamond\varphi]$
R4	From $\Box\varphi$ and $^{\Box}[\ldots]$ deduce $^{\Box}[\ldots\Box\varphi]$
R5	From $\Diamond\varphi$ and $^{\Box}[\ldots]$ deduce $^{\Box}[\ldots\Diamond\varphi]$

Examples of correct syntactic sequents: $\Box A \vdash A$; $A \vdash \Box\Diamond A$; $\Box A \vdash \Box\Box A$; $\Diamond A \vdash \Box\Diamond A$; $A \vdash \Diamond A$; $\Box[A \rightarrow B], \Diamond A \vdash \Diamond B$; $\vdash [\Box\neg A \rightarrow \neg\Box A]$; $\vdash [\neg\Diamond A \rightarrow \Box[A \rightarrow B]]$; $\vdash \Box[\Diamond\Box A \rightarrow \Box\Diamond A]$.

<u>Exercises on S5</u>

1. Translate the following sentences into wffs of S5:
   a. It's possible that John will win.
   b. You might be right.
   c. Oswald might have shot Kennedy.
   d. Mary can't possibly win if she drinks that.
   e. If John is a bachelor then he must be male.

2. Which of the following are correct semantic sequents? Provide counterexamples for those that are not.
   a. $\models \Box\Box[A \rightarrow A]$
   b. $\Box[A \vee B] \models [\Box A \vee \Box B]$
   c. $\Diamond\Diamond A \models \Diamond A$
   d. $\models [A \rightarrow \Box\Diamond A]$
   e. $\Box\Diamond A \models \Diamond\Box A$
   f. $\models \Box[\Box A \rightarrow \Box B] \vee \Box[\Box B \rightarrow \Box A]$
   g. $[A \rightarrow \Diamond B] \models [\Diamond B \rightarrow \neg A]$
   h. $[A \rightarrow B] \models \neg\Diamond[A \,\&\, \neg B]$
   i. $\neg\Diamond A \models \Box[A \rightarrow B]$
   j. $\models [\Box[\neg A \rightarrow A] \leftrightarrow \Box A]$
   k. $\models \Box[A \vee \Box B] \leftrightarrow [\Box A \vee \Box B]$

3. Prove the following by valid deduction:
   a. $[\Box A \,\&\, \Box B] \vdash \Box[A \,\&\, B]$
   b. $\Diamond[A \vee B] \vdash [\Diamond A \vee \Diamond B]$
   c. $\Box[A \rightarrow B] \vdash \Box[\Box A \rightarrow \Box B]$

d. $\Box[A \rightarrow B], \Box[\neg A \rightarrow B]] \vdash \Box B$
e. $\vdash [\Box\Diamond\Box A \leftrightarrow \Box A]$
f. $\neg\Box[A \rightarrow \Box\neg B] \vdash [\Box B \; \& \; \Diamond A]$
g. $\Box[A \rightarrow B], \Box[B \rightarrow C] \vdash \Box[A \rightarrow C]$

4. Translate the following arguments into sequents of S5 and decide whether or not the sequents are correct:
   a. It can’t be that John both did his homework and went to the movies. Since he went to the movies, he didn’t do his homework. (H, M)
   b. If it’s raining then it must be cloudy. If it’s cloudy then my dog is hiding under the couch. But my dog is not hiding under the couch, so it is not raining. (R, C, D)
   c. It’s impossible for me to run the gold coast marathon without doing a lot of training. So if I don’t do a lot of training then I can’t run the gold coast marathon. (R, T)

5. Translate and prove:
   a. Philby has to be either a spy or a counterspy or a mountebank. To be a counterspy he must betray his country. For him to be a spy or a mountebank necessarily implies daring on his part. Thus, Philby is necessarily either daring or a traitor. (S, C, M, T, D)
   b. Extensive services necessarily imply considerable funds. Funds are not necessarily considerable. Thus, if extensive services are necessary then, as a matter of necessity, the mayor is not necessarily competent. (S, F, M)
   c. Rain tomorrow is incompatible with our going on a picnic. It cannot fail to rain tomorrow. Therefore, it will be impossible for us to go on a picnic. (R, P)
   d. Necessarily, if there is a God at all, then there is necessarily a God. There is not necessarily no God. Hence there is necessarily a God. (G)
   e. It is not necessarily true that some statesmen are maniacs and a nuclear holocaust is unnecessary. Thus, a nuclear holocaust is necessary, for some statesmen are necessarily maniacs. (M, H)
   f. Rain tomorrow is incompatible with our going on a picnic. It cannot fail to rain tomorrow. Therefore, it will be impossible for us to go on a picnic. (R, P)
   g. It is possible that God exists. If He exists at all, then it necessarily follows that He exists necessarily. Therefore, God necessarily exists. (G).
   h. Necessarily, equality is compatible with separatism just in case integration is compatible with segregation. For equality strictly coimplies integration. And as for separatism and segregation, you can’t have one without the other. (E, P, I, S)