PHIL 331/MATH 281: Solutions for Week 2

The syntax of PC

b.i ‘Q’ is a wff, by FR1. So ‘¬Q’ is a wff, by FR2. ‘R’ is a wff, by FR1. So ‘(¬Q → R)’ is a wff, by FR3. ‘S’ is a wff, by FR1. So ‘((¬Q → R) ∨ S)’ is a wff, by FR3. ‘P’ is a wff, by FR1. So ‘(P ∧ ((¬Q → R) ∨ S))’ is a wff, by FR3. ■

c.ii Result: Between every pair of matched brackets in a wff there is a binary connective. *Proof*. We will prove by induction that the result is true for wffs with n connectives, for every $n \geq 0$. If φ is a wff with no connectives then φ is a sentence letter and has no brackets. So between every pair of matched brackets in φ (there are none) there is a binary connective. So the result is (vacuously) true for $n = 0$. Suppose that the result is true for $n = 0, 1, \ldots, k$. Suppose that φ is a wff with k+1 connectives. Since φ is a complex wff it either of the form ‘¬ψ’ or of the form ‘(ψ C ψ')’, for some wffs ψ and ψ' (with no more than k connectives, and so for which the result is true) and some binary connective C. If φ is of the first form, then any pair of matched brackets in φ is a pair of matched brackets in ψ, and so there is a binary connective between them. If φ is of the second form, then any pair of matched brackets is either a pair of matched brackets in ψ or a pair of matched brackets in ψ', in which case there is a binary connective between them, or it is the outermost pair of brackets, in which case there is the binary connective C between them. Either way, every pair of matched brackets in φ have a binary connective between them. So the result is true for $n = k + 1$. So the result is true for all n ≥ 0. ■ It now follows from this result that ‘(A → (A ∧ (¬(B))))’ is not a wff, because it contains a pair of matched brackets without a binary connective between them. ■

Unique decomposition

b.ii. ‘((A ∧ B) → ¬(A ∨ B))’

e. ‘A’ is a wff of length 1, so there is a wff of length 1. Any wff of length greater than 1 is a complex wff, and we can see from the formation rules that every complex wff has more than 3 symbols, so there are no wffs of length 2 or 3. If there is a wff φ of length n, then there is a wff ‘(¬φ)’ of length n+3. Since there is a wff of length 1, there is a wff of length 4, 7, 10, …. ‘(A ∧ A)’ is a wff of length 5, so there is a wff of length 5. So there is a wff of length of 8, 11, 14, …. ‘(A ∧ (A ∧ A))’ is a wff of length 9, so there is a wff of length 9. So there is a wff of length of 12, 15, 18, …. Finally, there is no wff of length 6, because such a wff could only be formed by combining a wff of length 1 with a wff of length 2, and there are no wffs of length 2. ■

f. We will prove by induction that the result is true for wffs with n connectives, for all $n \geq 0$. If φ is a wff with no connectives then φ is a sentence letter, so $s = 1$, $c = 0$, and $s = c + 1$. So the result is true for $n = 0$. Now suppose that the result is true for $n = 0, 1, \ldots, k$, and suppose that φ is a wff with $k + 1$ connectives. Since φ is a complex wff it is either of the form '$\neg\psi_1$' or of the form '$(\psi_1 \text{ C } \psi_2)$', for some wffs ψ_1 and ψ_2 (with no more than k connectives, so for which the result is true) and some binary connective C. Suppose that φ is of the first form. Let s_1 and c_1 be the number of occurrences of sentence letters and binary connectives in ψ_1. Since the result is true for ψ_1, $s_1 = c_1 + 1$. But $s = s_1$ and $c = c_1$, so $s = c + 1$. Suppose instead that φ is of the second form. Let s_1, c_1, s_2, and c_2 be the number of occurrences of sentence letters and binary connectives in ψ_1 and ψ_2 respectively. Since the result is true for ψ_1 and ψ_2, $s_1 = c_1 + 1$ and $s_2 = c_2 + 1$. So $(s_1 + s_2) = (c_1 + c_2 + 1) + 1$. But $s = s_1 + s_2$, and $c = c_1 + c_2 + 1$, so $s = c + 1$. Either way, $s = c + 1$, and the result is true for $n = k + 1$. So the result is true for every $n \geq 0$. ■