PHIL 331/MATH 281: Solutions for Week 2

Polish and reverse Polish notation

a.iii f('(C ∨ ((B ∧ ¬D) → C))') = '∨C→∧B¬DC'

a.iv f('(P ∧ (¬(Q → R) ∨ S))') = '∧P∨¬→QRS'

b.iii '→→AB→→BC→¬AC' = f('((A → B) → ((B → C) → (¬A → C)))')

b.iv '→∨¬→¬PQR↔∧P¬QP' = f('((¬(¬P → Q) ∨ R) → ((P ∧ ¬Q) ↔ P))')

c. We shall prove by induction that for each $n \geq 1$ if ε is an expression in Polish notation length n then ε is a wff iff (i) the sum of the symbols in ε is -1, and (ii) the sum of the symbols in any proper initial segment of ε is non-negative.

Base case

Suppose that ε is an expression of length 1.

Suppose that ε is a wff. Then ε is a sentence letter. So (i) the sum of the symbols in ε is -1, and (ii) the sum of the symbols in any proper initial segment of ε is non-negative (there are none).

Now suppose (i) that the sum of the symbols in ε is -1, and (ii) that the sum of the symbols in any proper initial segment of ε is non-negative. Because of (i), ε must be a sentence letter. Thus ε is a wff.

So the result is true for expressions of length 1.

Inductive step

Suppose that the result is true for expressions up to length k, for some $k \geq 1$, and suppose that ε is an expression of length $k + 1$ (so the length of ε is at least 2).

Suppose that ε is a wff. We will show (i) that the sum of the symbols in ε is -1, and (ii) that the sum of the symbols in any proper initial segment of ε is non-negative.

Since the length of ε is greater than one ε is a complex wff, so either ε = '¬φ', for some wff φ, or ε = 'Cψχ', for some binary connective C and wffs ψ and χ.

Suppose that ε = '¬φ'.

Since φ is an expression of length no greater than k, the result is true of φ. Since the result is true of φ, and since φ is a wff, the sum of the symbols in φ is -1. So the sum of the symbols in ε is 0 + (-1), which is -1.

Now suppose that σ is a proper initial segment of ε. Then either σ = '$\neg$' or σ = '$\neg\varphi'$', where φ' is some proper initial segment of φ. In the first case, the sum of the symbols in σ is 0 and hence is non-negative. In the second case, since the result is true of φ, and since φ is a wff, the sum of the symbols in any proper initial segment of φ is non-negative. Since φ' is a proper initial segment of φ, the sum of the symbols in φ' is non-negative. Thus the sum of the symbols in σ is non-negative (adding 0 to a non-negative number yields a non-negative number). So the sum of the symbols in any proper initial segment of ε is non-negative.

Now suppose that ε = '$C\psi\chi$'.

Since ψ and χ are expressions of length no greater than k, the result is true of ψ and χ. Since they are both wffs, the sum of the symbols in ψ is -1, and the sum of the symbols in χ is -1. So the sum of the symbols in ε is 1 + (-1) + (-1), which is -1.

Now suppose that σ is a proper initial segment of ε. Then either σ = C, or σ = '$C\psi'$', where ψ' is some proper initial segment of ψ, or σ = '$C\psi$', or σ = '$C\psi\chi'$', where χ' is some proper initial segment of χ. In the first case, the sum of the symbols in σ is 1 and hence is non-negative. In the second case, the sum of the symbols in σ is 1 + (non-negative number), which is non-negative (similar reasoning to above). In the third case, the sum of the symbols in σ is 1 + (-1), which is non-negative (similar to above). In the fourth case, the sum of the symbols in σ is 1 + (-1) + (non-negative number), which is non-negative (similar to above). So the sum of the symbols in any proper initial segment of ε is non-negative.

Now suppose (i) that the sum of the symbols in ε is -1, and (ii) that the sum of the symbols in any proper initial segment of ε is non-negative. We will show that ε is a wff.

Since ε is an expression of length $k + 1$, ε = '$\omega\varepsilon'$' for some symbol ω and some expression ε' of length $k \geq 1$. Since ω is a proper initial segment of ε, the value of ω is non-negative, so either ω is '$\neg$' or ω is C, for some binary connective C.

Suppose that ω is '$\neg$'.

So ε = '$\neg\varepsilon'$'. Since the sum of the symbols in ε is -1, the sum of the symbols in ε' is (-1) – 0, which is -1.

Suppose that σ is a proper initial segment of ε'. Then '$\neg\sigma$' is a proper initial segment of ε. Since the sum of the symbols of any proper initial segment of ε is non-negative, the sum of the symbols of '$\neg\sigma$' is non-negative. Thus the sum of the symbols of σ is non-negative.

Since ε' is an expression of length no more than *k*, the result is true for ε'. Since (i) the sum of the symbols of ε' is -1, and (ii) the sum of the symbols of any proper initial segment of ε' is non-negative, ε' is a wff.

So ε = '¬ε'' for some wff ε'. So ε is a wff.

Now suppose that ω is C, for some binary connective C.

So ε = 'Cε''. Since the sum of the symbols in ε is -1, and since the value of C is 1, and since the value of a symbol is never less than -1, there must be an expression α_1 such that ε = '$C\alpha_1\alpha_2$', where the sum of the symbols in '$C\alpha_1$' is 0. In fact there must be a shortest such expression. Take α_1 to be the shortest such expression. Since the value of C is 1, the sum of the symbols in α_1 is -1.

Suppose that σ is a proper initial segment of α_1. Then 'Cσ' is a proper initial segment of ε. Thus the sum of the symbols in 'Cσ' must be non-negative. In fact, it must be positive, because we are taking α_1 to be the shortest expression such that the sum of the symbols in '$C\alpha_1$' is 0. So the sum of the symbols in σ is non-negative.

Since α_1 is an expression of length no more than *k*, the result is true for α_1. Since (i) the sum of the symbols of α_1 is -1, and (ii) the sum of the symbols of any proper initial segment of α_1 is non-negative, α_1 is a wff.

We will now show that α_2 is also a wff. Since the sum of the symbols in ε is -1, and the sum of the symbols in '$C\alpha_1$' is 0, the sum of the symbols in α_2 is -1.

Suppose that σ is a proper initial segment of α_2. Then '$C\alpha_1\sigma$' is a proper initial segment of ε. Thus the sum of the symbols in '$C\alpha_1\sigma$' is non-negative. Since the sum of the symbols in '$C\alpha_1$' is 0, the sum of the symbols in σ is non-negative.

Since α_2 is an expression of length no more than *k*, the result is true for α_2. Since (i) the sum of the symbols of α_2 is -1, and (ii) the sum of the symbols of any proper initial segment of α_2 is non-negative, α_2 is a wff.

Since ε = '$C\alpha_1\alpha_2$', for some binary connective C and wffs α_1 and α_2, ε is a wff. ■

Substitution

a.ii. '(P → (P ∧ Q))'('(P ∧ P)'/'Q') = '(P → (P ∧ (P ∧ P)))'

a.iii '¬(A ∧ (A ∧ A))'('(A → A)'/'A') = '¬((A → A) ∧ ((A → A) ∧ (A → A)))'

b. We shall prove by induction that for every $n \geq 0$ any substitution instance of a wff with n connectives is a wff.

Base case

Suppose that φ is a wff with 0 connectives. Then $\varphi = \Lambda$, for some sentence letter Λ. If φ' is a substitution instance of φ, then $\varphi' = \varphi(\chi/\Lambda)$, for some wff χ. But $\varphi(\chi/\Lambda) = \chi$, which is a wff. So φ' is a wff. So the result is true for $n = 0$.

Inductive step

Suppose that the result is true for $n = 0, 1, \ldots, k$, and suppose that φ is a wff with $k + 1$ connectives. Since φ has at least one connective, φ is a complex wff. So either φ = ‘$\neg\psi$’, for some wff ψ, or φ = ‘$(\psi \text{ C } \psi')$’, for some binary connective C and wffs ψ and ψ'. If φ' is a substitution instance of φ, then $\varphi' = \varphi(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m)$, for some sentence letters $\Lambda_1, \ldots \Lambda_m$ in φ, and some wffs $\chi_1, \ldots, \chi_m$.

If φ = ‘$\neg\psi$’ then φ' = ‘$\neg\psi(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m)$’. Since ψ is a wff with no more than k connectives the result is true for ψ. So $\psi(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m)$ is a wff, and so φ' is a wff.

If φ = ‘$(\psi \text{ C } \psi')$’ then φ' = ‘$(\psi(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m) \text{ C } \psi'(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m))$’. Since ψ and ψ' are wffs with no more than k connectives the result is true for ψ and ψ'. So $\psi(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m)$ and $\psi'(\chi_1/\Lambda_1, \ldots, \chi_m/\Lambda_m)$ are wffs, and so φ' is a wff.

In either case, φ' is a wff. So the result is true for $n = k + 1$. ■

The semantics of PC

b.i If ‘$(A \rightarrow B)$’ is true then ‘$((A \vee C) \rightarrow (B \vee C))$’ is true.

b.ii If ‘$(A \rightarrow B)$’ is true then ‘$((A \wedge C) \rightarrow (B \wedge C))$’ is true.

b.iii Even if ‘$(A \rightarrow B)$’ is true ‘$((\neg A \wedge B) \leftrightarrow (A \vee B))$’ could be true or false.

d.ii

(	(	A	↔	B	)	→	(	¬	A	∧	B	)	)
		T	T	T		F		F	T	F	T		
		T	F	F		T		F	T	F	F		
		F	F	T		T		T	F	T	T		
		F	T	F		F		T	F	F	F		