PHIL 331/MATH 281: Selected Answers for Week 5

Tautologies

b. We shall prove by induction that for each odd number $n \geq 1$, φ_n is a tautology.

Base case. φ_1 = '($\varphi_0 \rightarrow$ P)' = '(P $\rightarrow$ P)', which is a tautology. So the result is true for $n = 1$.

Inductive step. Suppose the result is true for $n = k$, for some odd number k. Now, φ_{k+2} = '($\varphi_{k+1} \rightarrow$ P)' = '(($\varphi_k \rightarrow$ P) $\rightarrow$ P)'. Since the result is true for $n = k$, φ_k is a tautology, and hence true on every interpretation. So any interpretation on which 'P' is true is one on which φ_k is true, and hence one on which φ_{k+2} = '(($\varphi_k \rightarrow$ P) $\rightarrow$ P)' is true. And any interpretation on which 'P' is false is one on which φ_k is true, and hence one on which φ_{k+2} = '(($\varphi_k \rightarrow$ P) $\rightarrow$ P)' is true. So φ_{k+2} is a tautology. So the result is true for $n = k + 2$.

We shall now prove by induction that for each even number $n \geq 0$, φ_n is logically equivalent to 'P', and hence not a tautology.

Base case. φ_0 = 'P', which is logically equivalent to 'P'. So the result is true for $n = 0$.

Inductive step. Suppose the result is true for $n = k$, for some even number k. Now, φ_{k+2} = '($\varphi_{k+1} \rightarrow$ P)' = '(($\varphi_k \rightarrow$ P) $\rightarrow$ P)'. Since the result is true for $n = k$, φ_k is logically equivalent to 'P'. So any interpretation on which 'P' is true is one on which φ_k is true, and hence one on which φ_{k+2} = '(($\varphi_k \rightarrow$ P) $\rightarrow$ P)' is true. And any interpretation on which 'P' is false is one on which φ_k is false, and hence one on which φ_{k+2} = '(($\varphi_k \rightarrow$ P) $\rightarrow$ P)' is false. So φ_{k+2} is logically equivalent to 'P'. So the result is true for $n = k + 2$.

Putting together these two results we see that for every $n \geq 0$, φ_n is a tautology iff n is odd. ■

c.v. We shall prove by induction that for every $n \geq 0$, any wff with n connectives that is *special* (i.e. containing no sentence letter other than 'P' and no connective other than '$\leftrightarrow$') is either a tautology or logically equivalent to 'P'.

Base case. Suppose that φ has no connectives and is special. Then φ must be 'P', which is logically equivalent to 'P'. So the result is true for $n = 0$.

Inductive step. Suppose that the result is true for $n = 0, 1, \ldots, k$. Suppose that φ has $k + 1$ connectives and is special. Since φ has at least one connective, it is a complex wff. Since it is special, φ = '($\psi \leftrightarrow \chi$)', for some wffs ψ and χ which are also special. Since ψ and χ have no more than k connectives, the result is true for both ψ and χ, so each is either a tautology or logically equivalent to 'P'. Suppose that ψ and χ are both tautologies. Then every interpretation is one on which ψ and χ are both true, and hence one on which φ = '$(\psi \leftrightarrow \chi)$' is true. So φ is a tautology. Suppose that ψ is a tautology and χ is logically equivalent to 'P'. Then any interpretation on which 'P' is true is one on which ψ and χ are both true, and hence one on which φ = '$(\psi \leftrightarrow \chi)$' is true. And any interpretation on which 'P' is false is one on which ψ is true and χ is false, and hence one on which φ = '$(\psi \leftrightarrow \chi)$' is false. So φ is logically equivalent to 'P'. Similarly, if ψ is logically equivalent to 'P' and χ is a tautology then φ is logically equivalent to 'P'. Finally, suppose that ψ and χ are both logically equivalent to 'P'. Then any interpretation on which 'P' is true is one on which ψ and χ are both true, and hence one on which φ = '$(\psi \leftrightarrow \chi)$' is true. And any interpretation on which 'P' is false is one on which ψ and χ are both false, and hence one on which φ = '$(\psi \leftrightarrow \chi)$' is true. So φ is a tautology. In any case, φ is either a tautology or logically equivalent to 'P'. So the result is true for $n = k + 1$. ■