PHIL 331/MATH 281: Selected Answers for Week 6

Semantic entailment generalized

b. $\models \Delta$ iff every interpretation is one on which at least one wff in Δ is true; that is, iff every interpretation is one on which at least one wff in '$\neg\Delta$' is false; that is, iff there is no interpretation on which every wff in '$\neg\Delta$' is true; that is, iff '$\neg\Delta$' $\models$. So $\models \Delta$ iff '$\neg\Delta$' $\models$. ■

Some properties of entailment

a.iv. Suppose that $\Gamma, \varphi \models \psi$ and Γ, '$\neg\varphi$' $\models \psi$. Suppose that it is not the case that $\Gamma \models \psi$. Then there is some interpretation I on which every wff in Γ is true but ψ is false. But on I either φ is true or φ is false. So on I either φ is true or '$\neg\varphi$' is true. So on I either every wff in $\Gamma \cup \{\varphi\}$ is true, or every wff in $\Gamma \cup \{$'$\neg\varphi$'$\}$ is true. In the first case, since $\Gamma, \varphi \models \psi$ we have that ψ is true on I. In the second case, since Γ, '$\neg\varphi$' $\models \psi$ we also have that ψ is true on I. So in either case ψ is true on I. But that contradicts the result that ψ is false on I. So the supposition that it is not the case that $\Gamma \models \psi$ has led to a contradiction. So $\Gamma \models \psi$. So if $\Gamma, \varphi \models \psi$ and Γ, '$\neg\varphi$' $\models \psi$ then $\Gamma \models \psi$. ■

c. We will prove that for any set of wffs Γ, if (i) is true then (ii) is true, if (ii) is true then (iii) is true, and if (iii) is true then (i) is true, from which it follows that (i), (ii), and (iii) are equivalent.

If (i) is true then (ii) is true

Suppose that (i) is true. So Γ is consistent. So there is an interpretation I on which every wff in Γ is true. Suppose that (ii) is false. So for some wff φ we have that both $\Gamma \models \varphi$ and $\Gamma \models$ '$\neg\varphi$'. Since every wff in Γ is true on I, and since $\Gamma \models \varphi$, we have that φ is true on I. And since every wff in Γ is true on I, and since $\Gamma \models$ '$\neg\varphi$', we have that '$\neg\varphi$' is true on I, so that φ is false on I. But φ cannot be both true and false on I. So there is no such φ. So (ii) is true.

If (ii) is true then (iii) is true

Suppose that (ii) is true. So there is no wff φ such that both $\Gamma \models \varphi$ and $\Gamma \models$ '$\neg\varphi$'. Suppose that (iii) is false. So there is no wff φ such that it is not the case that $\Gamma \models \varphi$. So for every wff φ we have that $\Gamma \models \varphi$. In particular, we have that both $\Gamma \models$ 'A' and $\Gamma \models$ '$\neg$A'. So there is some wff φ such that both $\Gamma \models \varphi$ and $\Gamma \models$ '$\neg\varphi$', which contradicts the result above. So (iii) is true.

If (iii) is true then (i) is true

Suppose that (iii) is true. So there is some wff φ such that it is not the case that $\Gamma \models \varphi$. So there is some wff φ such that there is an interpretation on which every wff in Γ is true and φ is false. So there is an interpretation on which every wff in Γ is true. So Γ is consistent. So (i) is true. ■

Compactness

a. We will prove that if (1a) is true then (1b) is true, and if (1b) is true then (1a) is true, from which it follows that (1a) and (1b) are equivalent. Since (1c) is the contrapositive of (1b), it follows that (1a), (1b) and (1c) are all equivalent.

If (1a) is true then (1b) is true

Suppose that (1a) is true. So for all Γ and φ: if $\Gamma \models \varphi$ then $\Gamma' \models \varphi$ for some finite subset Γ' of Γ. Suppose that Γ is inconsistent. Since the empty set is consistent, Γ must be non-empty. Suppose that $\psi \in \Gamma$. Since Γ is inconsistent, Γ entails any wff. In particular, $\Gamma \models$ '$\neg\psi$'. So $\Gamma' \models$ '$\neg\psi$', for some finite subset Γ' of Γ (by 1(a)). So $\Gamma', \psi \models$. So $\Gamma' \cup \psi$ is inconsistent. But since Γ' is a finite subset of Γ and since $\psi \in \Gamma$, $\Gamma' \cup \psi$ is a finite subset of Γ. So some finite subset of Γ is inconsistent. So (1b) is true.

If (1b) is true then (1a) is true

Suppose that (1b) is true. So for all Γ: if Γ is inconsistent then some finite subset of Γ is inconsistent. Suppose that for some Γ and φ we have that $\Gamma \models \varphi$. So Γ, '$\neg\varphi$' $\models$. So $\Gamma \cup$ '$\neg\varphi$' is inconsistent. So some finite subset Γ' of $\Gamma \cup$ '$\neg\varphi$' is inconsistent. Suppose that Γ' does not contain '$\neg\varphi$'. Then Γ' is a subset of Γ. And since Γ' is inconsistent we have that $\Gamma' \models \varphi$. Suppose that Γ' does contain '$\neg\varphi$'. Then $\Gamma' = \Gamma'' \cup \{$'$\neg\varphi$'$\}$, for some finite subset Γ'' of Γ. Since Γ' is inconsistent, $\Gamma'' \cup \{$'$\neg\varphi$'$\}$ is inconsistent. So Γ'', '$\neg\varphi$' $\models$. So $\Gamma'' \models \varphi$. Either way, some finite subset of Γ entails φ. So (1a) is true. ■