PHIL 331/MATH 281: Week 1

Introduction

1. English is a natural language. We will be studying two artificial languages: **propositional calculus** (**PC**) (or: *propositional logic*, *sentential logic*) and **quantificational calculus** (**QC**) (or: *predicate logic*).

2. Why? Because the sentences of PC and QC are unambiguous and have precisely defined truth-conditions. So by translating our claims and arguments from English into PC or QC we can get clearer about the conditions under which those claims are true, and about whether or not those arguments are valid.

3. PC and QC are also interesting objects of study in their own right, and can perhaps shed light on the operation of natural languages.

Sets

1. For any things there is such a thing as the **set** of those things. So for Ithaca and Oxford there is a such a thing as the set of Ithaca and Oxford, written {Ithaca, Oxford}.

2. Each of the things is a **member** (or: *element*) of the set. So Ithaca is a member of {Ithaca, Oxford}, written Ithaca $\in$ {Ithaca, Oxford}. Nothing else is a member of the set. So Sydney is not a member of {Ithaca, Oxford}, written Sydney $\notin$ {Ithaca, Oxford}.

3. Two sets A and B are **identical** iff they have the same members: for all *x*, $x \in$ A iff $x \in$ B; equivalently: iff there is no *x* such that $x \in$ A and $x \notin$ B, or $x \in$ B and $x \notin$ A. So {Ithaca, Oxford}, {Oxford, Ithaca} and {Ithaca, Ithaca, Oxford, Oxford, Oxford} are all identical. These are three different ways of referring to the same set.

4. There is also such a thing as the **empty set**, written '{}' or '∅' – it has no members. There is exactly one empty set (*Proof*: suppose that A and B are both empty sets; since there is no *x* such that $x \in$ A, there is no *x* such that $x \in$ A and $x \notin$ B, and since there is no *x* such that $x \in$ B, so there is no *x* such that $x \in$ B and $x \notin$ A; so A and B are identical; so there is only one empty set. ■)

5. There are at least two ways to refer to a set: list its members, as in '{Ithaca, Oxford}', or specify a rule for being one of its members, as in '{*x*: *x* is a prime number}'.

   There are typically many different ways of referring to the same set: {Ithaca, Oxford} = {the town in which Wylie spent thanksgiving 2007, the city in which Wylie spent Christmas 2006}; {*x*: *x* is an even number} = {*x*: *x* + 1 is an odd number}.

   Some rules give rise to paradox. Example (Russell's paradox): Some sets are members of themselves: {*x*: *x* is not the Eiffel tower}; some sets are not members of themselves: {Ithaca, Oxford}; let S = {*x*: *x* is not a member of itself}; then is S a member of itself or not? One possible moral is that this rule cannot be used to define a set. Another is that we need a hierarchy of types of sets.

6. An important set: N = {0, 1, 2, …} (the **natural numbers**). Note that this is an infinite set.

7. Set A is said to be a **subset** of set B, written $A \subseteq B$, iff every member of A is a member of B: for all *x*, if $x \in A$ then $x \in B$; equivalently: iff there is no *x* such that $x \in A$ and $x \notin B$. A is a **proper subset** of set B, written $A \subset B$, iff A is a subset of B and A is not identical to B. So {Ithaca} is a proper subset of {Ithaca, Oxford}.

8. Every set is a subset of itself. (*Proof*: suppose that A is a set; for all *x*, if $x \in A$ then $x \in A$; so $A \subseteq A$. ■). The empty set is a subset of every set. (*Proof*: suppose that A is a set; since there is no *x* such that $x \in \{\}$, there is no *x* such that $x \in \{\}$ and $x \notin A$; so $\{\} \subseteq A$. ■).

9. For any two sets A and B, the **union** of A and B, written $A \cup B$, is $\{x: x \in A \text{ or } x \in B\}$. So {Ithaca, Oxford} ∪ {Oxford, Wagga Wagga} = {Ithaca, Oxford, Wagga Wagga}.

10. For any two sets A and B, the **intersection** of A and B, written $A \cap B$, is $\{x: x \in A \text{ and } x \in B\}$. So {Ithaca, Oxford} ∩ {Oxford, Wagga Wagga} = {Oxford}.

11. Two sets are said to be **disjoint** just in case their intersection is empty.

12. Exercises

    a. How many distinct sets are listed below?
    {1, 2, 2, 3, 3}, {1, 2, 3}, {2, 1}, {3, 2, 1}, {1, 1, 2}, {1, 2, 3, 2, 1}, {1, 2}

    b. List all the subsets of {1, 2, 3}

    c. If A is a set with *n* members, for some $n \in N$, then how many subsets does A have? For each $k \in \{0, 1, \ldots, n\}$, how many subsets of A have *k* members?

    d. Find the following unions:
    i. {1, 4, 3} ∪ {2, 3, 5} ∪ {1, 2} ∪ {2, 6}
    ii. {Ithaca, Ithaca} ∪ {Ithaca} ∪ {} ∪ {Sydney}

    e. Find the following intersections:
    i. {1, 4, 3} ∩ {2, 3, 1, 5} ∩ {1, 2} ∩ {2, 1, 6}
    ii. {Ithaca} ∩ {Ithaca, Ithaca} ∩ {} ∩ {Ithaca, Ithaca, Ithaca}

Tuples

1. For any ordered things there is such a thing as the **tuple** (or: *sequence*) of those ordered things. So for Ithaca and Oxford, in that order, there is the tuple of Ithaca and Oxford, in

that order, written <Ithaca, Oxford>. And for Oxford and Ithaca, in that order, there is the tuple <Oxford, Ithaca>.

2. If there are *n* ordered things, then the tuple is said to be an ***n*-tuple** (or: to be a sequence of length *n*). A 2-tuple is called an **ordered pair**, or just a **pair**. So <Ithaca, Oxford> is an ordered pair. A 3-tuple is called an **ordered triple**, or just a **triple**.

3. An *n*-tuple has *n* **components** (or: *coordinates*). The first component of $<x_1, \ldots, x_n>$ is x_1, the second component is x_2, and so on.

4. Two tuples $<x_1, \ldots, x_n>$ and $<y_1, \ldots, y_m>$ are **identical** iff (i) $n = m$, and (ii) for each $1 \leq k \leq n$, $x_k = y_k$. So <Ithaca, Oxford>, <Oxford, Ithaca>, and <Ithaca, Ithaca, Oxford, Oxford, Oxford> are all distinct tuples.

5. There is such a thing as a 0-tuple: <>. There is exactly one 0-tuple. There are also 1-tuples: <Ithaca>, <2>, <'a'>, and so on.

6. Exercises

   a. How many distinct tuples are listed below?
   $<1, 4>, <1, 4, 1>, <1^2, 2^2>, <4, 1>, <1, 1, 4>, <1, 2^2, 1>$

   b. List all the 3-tuples whose components are members of {1, 2}.

   c. If A is a set with *n* members, for some $n \in$ N, and $k \in$ N, then how many *k*-tuples are there whose components are all members of A?

Functions

1. A **function** is a rule which maps each thing in one set, its **domain**, onto things in another set, its **codomain**.

2. If *f* is a function whose domain is A and whose codomain is B then we say that *f* maps A into B, or that *f* is a function from A to B. We write f: A $\rightarrow$ B.

3. If *f* maps $a \in$ A onto $b \in$ B then we say that when the **argument** of *f* is *a*, the **value** of *f* is *b*, and we write $f(a) = b$, or f: $a \mapsto b$.

   Examples: Let *Cap* be a function from the set of US states to the set of US cities such that *Cap*(*x*) is the capital of *x*. Let *Stat* be a function from the set of US cities to the set of US states such that *Stat*(*x*) is the state that *x* is in.

   Cap(Idaho) =
   Stat(Montgomery) =

4. The **range** of a function f: A → B is the set of things which it takes as values: $\{x: x = f(y)$, for some $y\}$. We sometimes write this as f(A). Its range is a subset of its codomain (perhaps proper, perhaps not).

The range of *Cap* is:
The range of *Stat* is:

5. If the arguments of f are ordered pairs, then we can think of f as a **2-place function**. Rather than writing $f(<x, y>) = z$, we write $f(x, y) = z$. Similarly, if the arguments of f are n-tuples, then we can think of f as an ***n*-place function**. Rather than writing $f(<x_1, \ldots, x_n>) = z$, we write $f(x_1, \ldots, x_n) = z$.

6. We will be especially interested in **truth functions**: an n-place truth function is a function of n arguments, each of which is a truth value (T or F), and whose value is a truth value. We can represent truth functions using **truth tables**. Examples:

a.

x	$f(x)$
T	T
F	T

b.

x_1	x_2	$g(x_1, x_2)$
T	T	F
T	F	T
F	T	T
F	F	T

c.

x_1	x_2	x_3	$h(x_1, x_2, x_3)$
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

7. A function f: A → B is said to be **1-1** (or: *injective*) just in case for all x and y in A: if $f(x) = f(y)$ then $x = y$.

Cap is:
Stat is:

f is said to be **onto** (or: *surjective*) just in case for all $y \in$ B there is some $x \in$ A such that $f(x) = y$. That is, f(A) = B.

Cap is:
Stat is:

f is said to be **1-1 and onto** (or: *bijective*) iff it is 1-1 and onto.

8. If there is a bijection between the set of natural numbers {0, 1, 2, …} and the set A then A is said to be **denumerable**.

The set of rational numbers is denumerable, but the set of real numbers is not. Both sets are infinite, but the latter is more infinite.

Any set which is either finite or denumerable is said to be **countable**.

9. If f: A → B and g: C → D are functions such that B ⊆ C (i.e. the codomain of f is a subset of the domain of g), then we can **compose** f and g to get a function h: A → D such that $h(x) = g(f(x))$. We say that $h = g(f)$ or that $h = g \circ f$.

   Cap∘*Stat*(Miami) =
   Stat∘*Cap*(California) =

10. Exercises

   a. How many of the following are there?
      i. 1-place truth functions
      ii. 2-place truth functions
      iii. 3-place truth functions
      iv. *n*-place truth functions

   b. Find the range of the following functions:
      i. f: N → N, $f(x) = 2x + 1$
      ii. f: {x: x is an Australian state} → {x: x is an Australian city}, $f(x)$ = the capital of x
      iii. f: {Ithaca} → {T, F}, $f(x)$ = T just in case x is gorgeous

   c. Classify the following functions as 1-1, onto, or bijective:
      i. f: N → N, $f(x) = x^2$
      ii. f: N → {1, 3, 5, …}, $f(x) = 2x + 1$
      iii. f: {x: x is an English word} → {x: x is an English letter}, $f(x)$ = the first letter of x.

   d. Compose the following functions:
      i. f: N → N, $f(x) = x^2$; and g: N → N, $g(x) = 2x + 1$
      ii. f: {x: x is an English word} → {x: x is a set of English letters}, $f(x)$ = the set of letters in x; and g: {x: x is a set of English letters} → {x: x is a set of English words}, $g(x)$ = {y: y is a word whose letters are members of x}.