

PHIL 331/MATH 281: Week 10

Five key results

1. If ' $\forall v\phi$ ' is true on an interpretation I then $\phi(\kappa/v)$ is true on I, for any individual constant κ . (But not conversely.)

Proof. Suppose that ' $\forall v\phi$ ' is true on I. So for every assignment α , ' $\forall v\phi$ ' is true on I relative to α . So for every assignment α , and for every v -variant α' of α , ϕ is true on I relative to α' . So for every assignment α , and for any individual constant κ (no matter what κ denotes on I), $\phi(\kappa/v)$ is true on I relative to α . That is, $\phi(\kappa/v)$ is true on I. ■

2. If ' $\exists v\phi$ ' is true on an interpretation I, and κ is an individual constant that does not occur in ' $\exists v\phi$ ', then there is an interpretation I' such that ' $\exists v\phi$ ' and $\phi(\kappa/v)$ are both true on I'.

Proof. Suppose that ' $\exists v\phi$ ' is true on I. So for every assignment α , ' $\exists v\phi$ ' is true on I relative to α . So for every assignment α , and for some v -variant α' of α , ϕ is true on I relative to α' . Let κ be any individual constant that does not occur in ' $\exists v\phi$ '. Let I' be an interpretation that is the same as I except it assigns $\alpha'(v)$ to κ . Since κ does not occur in ' $\exists v\phi$ ', ' $\exists v\phi$ ' is still true on I'. And for every assignment α , $\phi(\kappa/v)$ is true on I' relative to α . That is, $\phi(\kappa/v)$ is true on I'. ■

3. ' $\neg\forall v\phi$ ' is true on an interpretation I iff ' $\exists v\neg\phi$ ' is true on I.

Proof. Suppose that ' $\neg\forall v\phi$ ' is true on an interpretation I. So for every assignment α , ' $\neg\forall v\phi$ ' is true on I relative to α . So for every assignment α , ' $\forall v\phi$ ' is false on I relative to α . So for every assignment α , there is a v -variant α' of α such that ϕ is false on I relative to α' . So for every assignment α , there is a v -variant α' of α such that ' $\neg\phi$ ' is true on I relative to α' . So for every assignment α , ' $\exists v\neg\phi$ ' is true on I relative to α . So ' $\exists v\neg\phi$ ' is true on I. The proof in the other direction is similar. ■

4. ' $\neg\exists v\phi$ ' is true on an interpretation I iff ' $\forall v\neg\phi$ ' is true on I.

Proof. Similar to the previous. ■

5. If ' $\kappa = \kappa'$ ' is true on an interpretation I then ϕ is true on I iff $\phi(\kappa'/\kappa)$ is true on I, and ϕ is true on I iff $\phi(\kappa/\kappa')$ is true on I.

Proof. Suppose that ' $\kappa = \kappa'$ ' is true on an interpretation I. So κ and κ' denote the same thing on I. So ϕ is true on I iff $\phi(\kappa'/\kappa)$ is true on I. And ϕ is true on I iff $\phi(\kappa/\kappa')$ is true on I. ■

6. Exercises

- a. Complete the proof of the third result.

- b. Prove the fourth result.
- c. Prove that the converse of the first result does not hold by finding an interpretation I and a wff ϕ such that $\phi(\kappa/x)$ is true on I for every individual constant κ , but ' $\forall x\phi$ ' is false on I .

Models and consistency

1. If Γ is a set of wffs and I is an interpretation such that every wff in Γ is true on I then we say that I is a **model** of Γ . If Γ has a model then we say that Γ is **consistent**. Otherwise we say that Γ is **inconsistent**, and we write $\Gamma \not\models$. If Γ contains just one member ϕ then rather than talk about $\{\phi\}$ being consistent or inconsistent we talk about ϕ itself being consistent or inconsistent.
2. Any model whose domain is a finite set is called a **finite model**. Otherwise it is called an **infinite model**.
3. We can *prove* that a wff or set of wffs is consistent by giving a model of the wff or wffs.

Example: $\{ 'Fa', '¬Fb', '∀x(Fx → ∃yGxy)' \}$ is consistent. *Proof.* Let I be the following interpretation: the domain is $\{1, 2\}$; 'a' denotes 1; 'b' denotes 2; 'F' denotes $\{1\}$; 'G' denotes $\{ \langle 1, 1 \rangle \}$. Then I is a model of this set of wffs. So the set is consistent. ■

4. We can prove that a wff or set of wffs is *inconsistent* by arguing that it has no model.

Example: $\{ 'Fa', '∀x(Fx → Gx)', '(Ga → ¬∃xFx)' \}$ is inconsistent. *Proof.* Suppose that there is an interpretation I on which every wff in this set is true. Since ' $∀x(Fx → Gx)$ ' is true on I , ' $(Fa → Ga)$ ' is true on I (by the first key result). Since ' Fa ' and ' $(Fa → Ga)$ ' are true on I , ' Ga ' is true on I . Since ' Ga ' and ' $(Ga → ¬∃xFx)$ ' are true on I , ' $¬∃xFx$ ' is true on I . Since ' $¬∃xFx$ ' is true on I , ' $∀x¬Fx$ ' is true on I (by the fourth key result). Since ' $∀x¬Fx$ ' is true on I , ' $¬Fa$ ' is true on I (by the first key result). Since ' $¬Fa$ ' is true on I , ' Fa ' is false on I . But ' Fa ' is true on I , which is a contradiction. So there is no such I , and the set is inconsistent. ■

5. Last time we saw that for any given wff it is impossible to make a list of all the possible interpretations of the wff, even in principle (because there are uncountably many possible domains for the interpretation). But we also saw that for any given finite domain it *is* possible to list all the possible interpretations of the wff, and thus to make a truth table for the wff relative to that domain. And we get the same column of truth values under the wff relative to *any* finite domain of the same size. Thus, given that a wff has a finite model then we have an effective procedure for finding one: start with the domain $\{1\}$; construct a truth table for the wff; if there is a row of the table in which the wff is true then that row is a model for the wff and we can stop; otherwise repeat for the domain $\{1, 2\}$; and so on. (This can be extended to any finite set of wffs.)

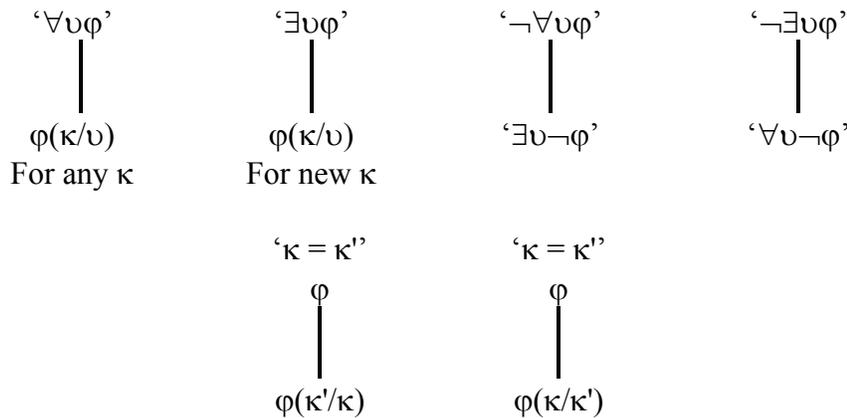
6. But this is not an effective procedure for deciding whether or not a wff has a finite model in the first place – if it does not, then the procedure will never stop. It is also unable to find infinite models. And it is in practice very laborious: the wff ‘ $(Fab \wedge \neg a = b)$ ’ has no model on a domain with one member, and on a domain with two members it has 64 possible interpretations relative to that domain. In the next section we will see a more efficient procedure for checking for consistency (but it is still not an *effective* one).

7. Exercises

- a. Prove that each of the following sets of wffs is consistent by finding a model:
- i. {‘ Fa ’, ‘ $(\neg a = b \wedge \neg b = c)$ ’, ‘ $\forall x(Fx \rightarrow x = c)$ ’}
 - ii. {‘ $\forall y\exists x(Fx \wedge Gy)$ ’, ‘ $\exists x\forall y(Fx \wedge Gy)$ ’}
 - iii. {‘ $\forall x\exists y(Py \wedge \neg x = y)$ ’, ‘ $\exists x\exists y((Px \wedge Py) \wedge \neg x = y)$ ’}
- b. Prove that each of the following sets of wffs is inconsistent by arguing that it has no model:
- i. {‘ $\exists x\forall y x = y$ ’, ‘ Fa ’, ‘ $\neg Fb$ ’}
 - ii. {‘ $\exists x\forall y x = y$ ’, ‘ $\forall x\exists y \neg x = y$ ’}
 - iii. {‘ $\exists xRxx$ ’, ‘ $\neg a = b$ ’, ‘ $\forall x\forall y(Rxy \rightarrow ((x = a \wedge y = b) \vee (x = b \wedge y = a)))$ ’}
- c. In PC we proved that if Δ and Γ are sets of wffs such that $\Delta \subseteq \Gamma$ then (i) if Γ is consistent then Δ is consistent, and (ii) if Δ is inconsistent then Γ is inconsistent. Do these results hold for QC as well?

Extending the tableau system to QC

1. For QC we add the following six development rules to the tableau system (so in QC we have 15 development rules in total):



2. We also add a new branch closing rule (so in QC we have 2 branch closing rules in total): a branch is closed if it contains a wff of the form ‘ $\neg\kappa = \kappa$ ’, for some individual constant κ .

3. We also need to extend the definition of what it is for a wff to be fully developed on a branch. A wff of the form $\exists u\phi$ is fully developed on a branch just in case the \exists rule has been applied to it at least once on the branch. A wff of the form $\forall u\phi$ is fully developed on a branch just in case the \forall rule has been applied to it for every individual constant on the branch. A wff of the form $\kappa = \kappa'$ is fully developed on a branch just in case the first '=' rule has been applied to it for every wff on the branch that contains κ and the second '=' rule has been applied to it for every one that contains κ' .
4. Note that for a wff of the form $\forall u\phi$ on a branch we may need to apply the \forall rule to it multiple times on that branch for it to be fully developed on that branch. So too with a wff of the form $\kappa = \kappa'$ and the '=' rules. For any other wff on a branch we need only apply a development rule to it once.
5. This means that sometimes a wff needs to have a rule applied to it infinitely many times for it to be fully developed on a branch, and thus that sometimes a branch needs to be infinite to be fully developed. There is no problem with this principle, but it means that the tableau technique no longer gives us an effective procedure for deciding whether or not a given set of wffs is consistent.
6. Result: If a tableau has no open branches then its root wffs are inconsistent.

Proof. We modify the proof from PC. We will first establish the preliminary result that if τ is a tableau with at least one consistent branch (i.e. a branch whose wffs are jointly consistent), and if τ' is a tableau obtained from τ by the application of a development rule to a developable wff ϕ on a branch β of τ , then τ' also has at least one consistent branch. For suppose that τ has a consistent branch. Suppose that it has a consistent branch that is distinct from β . Since this branch remains unchanged when the development rule is applied to ϕ on β , it is also a branch of τ' . So τ' has at least one consistent branch. Now suppose that β is the only consistent branch of τ . Since ϕ is developable it has one of the following forms: $\neg\neg\psi$, $(\psi \wedge \chi)$, $(\psi \vee \chi)$, $(\psi \rightarrow \chi)$, $(\psi \leftrightarrow \chi)$, $\neg(\psi \wedge \chi)$, $\neg(\psi \vee \chi)$, $\neg(\psi \rightarrow \chi)$, $\neg(\psi \leftrightarrow \chi)$, $\forall u\psi$, $\exists u\psi$, $\neg\forall u\psi$, $\neg\exists u\psi$, or $\kappa = \kappa'$. The first nine cases were dealt with in PC, so we need only consider the last five cases. Suppose that $\phi = \forall u\psi$, for some wff ψ . Then applying the appropriate development rule to ϕ on β yields a branch $\beta' = \beta + \psi(\kappa/u)$, for some individual constant κ . Since β is consistent there is an interpretation on which all the wffs in β are true. So it is an interpretation on which $\phi = \forall u\psi$ is true. So it is an interpretation on which $\psi(\kappa/u)$ is true (by the first key result). So it is an interpretation on which all the wffs in β' are true. So β' is a consistent branch. Suppose that $\phi = \exists u\psi$, for some wff ψ . Then applying the appropriate development rule to ϕ on β yields a branch $\beta' = \beta + \psi(\kappa/u)$, for some constant κ which does not appear on β . Since β is consistent there is an interpretation on which all the wffs in β are true. So it is an interpretation on which $\phi = \exists u\psi$ is true. Since κ does not appear on β there is an interpretation I' such that every wff on β and $\psi(\kappa/u)$ is true on I' (by the second key result). So I' is a model of β' , and β' is a consistent branch. Suppose that $\phi = \neg\forall u\psi$, for some wff ψ . Then applying the $\neg\forall$ development rule to ϕ on β yields a branch $\beta' = \beta$

+ $\exists v \neg \psi$ '. Since β is a consistent branch there is an interpretation I such that every wff in β is true on I . So $\varphi = \neg \forall v \psi$ is true on I . So $\exists v \neg \psi$ is true on I (by the third key result). So every wff in β' is true on I . So β' is a consistent branch. Suppose that $\varphi = \neg \exists v \psi$, for some wff ψ . By similar reasoning to the previous case, applying the $\neg \exists$ development rule to φ on β yields a consistent branch. Suppose that $\varphi = \kappa = \kappa'$, for some individual constants κ and κ' , and suppose that ψ is a wff on β that contains κ or κ' . Then applying the appropriate '=' development rule to φ and ψ on β yields either a branch $\beta' = \beta + \psi(\kappa/\kappa')$ or a branch $\beta' = \beta + \psi(\kappa'/\kappa)$. Since β is a consistent branch there is an interpretation I such that every wff in β is true on I . So $\varphi = \kappa = \kappa'$ and ψ are both true on I . So $\psi(\kappa/\kappa')$ and $\psi(\kappa'/\kappa)$ are both true on I (by the fifth key result). So every wff in β' is true on I . So β' is a consistent branch. This establishes the preliminary result. Now suppose that τ is a tableau with no open branches. This means that every branch is closed. This means that every branch is inconsistent. For if a branch is closed then either it contains a pair of wffs φ and $\neg \varphi$, and there is no interpretation on which these wffs are both true, or it contains a wff $\neg \kappa = \kappa$, and there is no interpretation on which this wff is true. So τ has no consistent branches. But that means that its root wffs must be inconsistent. For τ has been developed from its root wffs by the application of zero or more development rules. By the preliminary result above if these root wffs were consistent then applying a development rule would always yield a tableau with at least one consistent branch. But τ has no consistent branches. So its root wffs are inconsistent.

■

7. So if we can produce a tableau whose root wffs are the members of Γ and which has no open branches then we can conclude that Γ is inconsistent (the branches need not be fully developed).

Example: $\{ 'Fa', \forall x(Fx \rightarrow Gx), (Ga \rightarrow \neg \exists x Fx) \}$

8. Result: If a tableau has a fully developed open branch then its root wffs are consistent.

Proof. Again, we modify the proof from PC. Suppose that we have a tableau with a fully developed open branch β . Define an interpretation I of the language as follows: for each individual constant κ of QC, if κ appears in any wff on β then add a new member to the domain of I and let κ denote that member, unless $\kappa = \kappa'$ appears on β and κ' has already been interpreted, in which case let κ denote the same thing as κ' ; let every other constant denote some member of the domain (it doesn't matter which); for each sentence letter Λ of QC, let Λ denote T iff Λ is on β ; for each n -place predicate letter Π of QC and each n -tuple of individual constants $\langle \kappa_1, \dots, \kappa_n \rangle$ of QC, let $\langle |\kappa_1|, \dots, |\kappa_n| \rangle$ be in the denotation of Π iff $\Pi \kappa_1 \dots \kappa_n$ is on β . We shall now establish the preliminary result that every wff on β is true on I . We shall proceed by induction on the number n of connectives in the wffs on β . Suppose that φ is a wff on β with no connectives. Then either $\varphi = \Lambda$, for some sentence letter Λ , or $\varphi = \Pi \kappa_1 \dots \kappa_n$, for some n -place predicate Π and individual constants $\kappa_1, \dots, \kappa_n$, or $\varphi = \kappa = \kappa'$, for some individual constants κ and κ' . In each case φ is true on I , by the definition of I . So every wff on β that has no connectives is true on the interpretation I . So the result is true for $n = 0$. Now suppose

the result is true for $n = 0, 1, \dots, k$, for some $k \geq 0$, and that φ is a wff on β with $k + 1$ connectives. Since φ has at least one connective it has one of the following forms: ' $\neg\psi$ ', ' $(\psi \wedge \chi)$ ', ' $(\psi \vee \chi)$ ', ' $(\psi \rightarrow \chi)$ ', ' $(\psi \leftrightarrow \chi)$ ', ' $\forall v\psi$ ', ' $\exists v\psi$ '. The first five cases are handled in just the same way as they were in PC, so we only need to consider the last two cases here. So suppose that $\varphi = \forall v\psi$, for some variable v and wff ψ . Since φ is a developable wff and β is a fully developed branch, $\psi(\kappa/v)$ is on β , for every individual constant κ on β . Since $\psi(\kappa/v)$ is a wff on β that has fewer connectives than φ , the result is true for $\psi(\kappa/v)$. So $\psi(\kappa/v)$ is true on I for every individual constant κ on β . So ψ is true on I no matter what member of the domain is assigned to v . So ' $\forall v\psi$ ' is true on I. That is, φ is true on I. Suppose instead that $\varphi = \exists v\psi$, for some variable v and wff ψ . Since φ is a developable wff and β is a fully developed branch, $\psi(\kappa/v)$ is on β , for some individual constant κ on β . Since $\psi(\kappa/v)$ is a wff on β that has fewer connectives than φ , the result is true for $\psi(\kappa/v)$. So $\psi(\kappa/v)$ is true on I for some individual constant κ on β . So ψ is true on I for when at least one member of the domain is assigned to v . So ' $\exists v\psi$ ' is true on I. That is, φ is true on I. So the result is true for $n = k + 1$. This establishes the preliminary result. We thus have an interpretation I such that every wff in β is true on I. But β contains the root wffs of the tableau, so every root wff of the tableau are true on I. So the root wffs of the tableau are consistent. ■

9. So if we can produce a tableau whose root wffs are the members of Γ and which has a fully developed open branch then we can conclude that Γ is consistent. Moreover, from this open branch we can read off a model for Γ .

Example: $\{\exists x(Fx \rightarrow Gx), \neg(\exists xFx \rightarrow \exists xGx)\}$

10. Some more examples:

$\{\forall x\forall y\forall z((Fxy \wedge Fyz) \rightarrow Fxz), \forall x\forall y(Fxy \rightarrow Fyx), \neg\forall xFxx\}$
 $\forall x\exists yFxy$
 $\{Fa, \neg Fb, \forall x(Fx \rightarrow \exists yGxy)\}$

11. Exercises

Use the tableau technique to decide (where possible) whether or not the following sets of wffs are consistent:

- $\{\exists x(Fx \vee Gx), (\exists xFx \vee \exists xGx)\}$
- $\{\exists x\forall yFxy, \neg\forall y\exists xFxy\}$
- $\{\forall y\exists x(Fx \wedge Gy), \exists x\forall y(Fx \wedge Gy)\}$
- $\neg\exists x(Fx \rightarrow \forall xFx)$
- $\{\forall x\forall y x = y, \exists xPx, \forall xPx\}$
- $\{\forall x\forall y x = y, \exists xPx, \neg\forall xPx\}$
- $\{\forall x\forall y(Rxy \rightarrow \neg Ryx), \neg\forall x\neg Rxx\}$
- $\{\forall x\exists y(Py \wedge \neg x = y), \exists x\exists y((Px \wedge Py) \wedge \neg x = y)\}$
- $\{\exists x\neg Rxx, \forall x\exists yRxy, \forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz)\}$
- $\{\forall x\exists y(Fx \rightarrow Gy), \neg\exists y\forall x(Fx \rightarrow Gy)\}$

Soundness and completeness

1. As for PC, if Γ is a set of wffs (possibly infinite) say that Γ is **syntactically inconsistent**, written $\Gamma \vdash$, iff there is a closed tableau whose root wffs are the wffs in Γ .
2. Then we have the following results about the tableau system for QC (the proofs are exactly the same as for PC):

Soundness: if $\Gamma \vdash$ then $\Gamma \models$

Completeness: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$