PHIL 331/MATH 281: Week 11

Logical validity

1. A closed wff φ of QC is said to be **logically valid** just in case it is true on every interpretation. We write '⊨ φ' to mean that φ is logically valid.

2. We can prove that a closed wff is *not* logically valid by giving an interpretation on which it is false.

   Example: '(∃*x*F*x* → Fa)' is not logically valid. *Proof*. Let I be the following interpretation: the domain is {1, 2}; 'a' denotes 2; 'F' denotes {1}. Then '∃*x*F*x*' is true on I and 'Fa' is false on I, so '(∃*x*F*x* → Fa)' is false on I. So there is an interpretation on which '(∃*x*F*x* → Fa)' is false. So it is not logically valid. ■

3. We can prove that a closed wff *is* logically valid by arguing that that there is no interpretation on which it is false.

   Example: '(Fa → ∃*x*F*x*)' is logically valid. *Proof*. Suppose that I is an interpretation on which it is false. So 'Fa' is true on I and '∃*x*F*x*' is false on I. But if 'Fa' is true on I then the thing that 'a' denotes is a member of the set that 'F' denotes. So there is something in the set that 'F' denotes. So '∃*x*F*x*' is true on I. But that contradicts the result that '∃*x*F*x*' is false on I. So there is no such interpretation, and '(Fa → ∃*x*F*x*)' is logically valid. ■

4. Result: a closed wff φ is logically valid iff '¬φ' in inconsistent. That is, ⊨ φ iff '¬φ' ⊨.

   Proof. φ is logically valid iff it is true on every interpretation. That is, iff '¬φ' is false on every interpretation. That is, iff there is no interpretation on which '¬φ' is true. That is, iff '¬φ' is inconsistent. ■

5. This result means that we can use the tableau method to check whether or not a closed wff φ is logically valid: if we can produce a tableau whose sole root wff is '¬φ' and which is closed, then we can conclude that '¬φ' is inconsistent, and hence that φ is logically valid; if we can produce a tableau whose sole root wff is '¬φ' and which has a fully developed open branch, then we can conclude that '¬φ' is consistent, and hence that φ is not logically valid.

   Examples: use tableaux to prove the two examples above.

6. Exercises

   a. For each of the following closed wffs either prove that it is logically valid or prove that it is not:
      i. '∀*x* *x* = *x*'

ii. '$(\neg\exists x \mathrm{F}x \vee \mathrm{Fa})$'
iii. '$\exists x(\mathrm{F}x \rightarrow \neg\mathrm{F}x)$'
iv. '$\forall x(\mathrm{P}x \rightarrow \forall y\mathrm{P}y)$'
v. '$\exists x(\mathrm{P}x \rightarrow \forall y\mathrm{P}y)$'
vi. '$(\forall x(\exists x\mathrm{F}x \rightarrow \mathrm{F}x) \rightarrow (\exists x\mathrm{F}x \rightarrow \forall x\mathrm{F}x))$'

b. Prove the following:
   i. If φ is closed wff then φ is logically valid iff '$\forall x\varphi$' is logically valid.
   ii. If φ is a wff whose only free variable is 'x' then '$(\forall x\varphi \rightarrow \varphi(\tau/x))$' is logically valid, provided τ is free for x in φ. Give an example to show that the proviso is necessary.

Logical Equivalence

1. Two closed wffs φ and ψ are said to be **logically equivalent** just in case they have the same truth value on every interpretation. We write '$\varphi \dashv\vDash \psi$' to mean that φ and ψ are logically equivalent.

2. We can prove that two closed wffs are *not* logically equivalent by finding an interpretation on which they have different truth values.

   Example: '$\exists x(\mathrm{F}x \rightarrow \mathrm{A})$' and '$(\exists x\mathrm{F}x \rightarrow \mathrm{A})$' are not logically equivalent. *Proof*. Let I be the following interpretation: the domain is {1, 2}; 'A' is false; 'F' denotes {1}. Then '$\exists x(\mathrm{F}x \rightarrow \mathrm{A})$' is true on I (because '$(\mathrm{F}x \rightarrow \mathrm{A})$' is true when '$x$' is assigned 2), but '$(\exists x\mathrm{F}x \rightarrow \mathrm{A})$' is false on I (because '$\exists x\mathrm{F}x$' is true on I and 'A' is false on I). So there is an interpretation on which '$\exists x(\mathrm{F}x \rightarrow \mathrm{A})$' and '$(\exists x\mathrm{F}x \rightarrow \mathrm{A})$' have different truth values, and they are not logically equivalent. ■

3. We can prove that two closed wffs *are* logically equivalent by arguing that there is no interpretation on which they have different truth values.

   Example: '$\forall x(\mathrm{F}x \wedge \mathrm{A})$' and '$(\forall x\mathrm{F}x \wedge \mathrm{A})$' are logically equivalent. *Proof*. '$\forall x(\mathrm{F}x \wedge \mathrm{A})$' is true on an interpretation I iff everything in the domain of I is such that: it is in the set denoted by 'F' on I, and 'A' is true on I. That is, iff 'A' is true on I, and everything in the domain is such that: it is in the set denoted by 'F' on I. That is, iff '$(\forall x\mathrm{F}x \wedge \mathrm{A})$' is true on I. ■

4. Result: closed wffs φ and ψ are logically equivalent iff '$(\varphi \leftrightarrow \psi)$' is logically valid. That is, $\varphi \dashv\vDash \psi$ iff $\vDash$ '$(\varphi \leftrightarrow \psi)$'.

   Proof. φ and ψ are logically equivalent iff φ and ψ have the same truth value on every interpretation. That is, iff '$(\varphi \leftrightarrow \psi)$' is true on every interpretation. That is, iff '$(\varphi \leftrightarrow \psi)$' is logically valid. ■

5. So we can use tableau to check whether or not closed wffs φ and ψ are logically equivalent: use tableau to check whether or not ‘(φ ↔ ψ)’ is logically valid; that is: use tableau to check whether or not ‘¬(φ ↔ ψ)’ is consistent. If we can produce a tableau whose sole root wff is ‘¬(φ ↔ ψ)’ and which is closed, then we can conclude that ‘¬(φ ↔ ψ)’ is inconsistent, and hence that ‘(φ ↔ ψ)’ is logically valid, and hence that φ and ψ are logically equivalent; if we can produce a tableau whose sole root wff is ‘¬(φ ↔ ψ)’ and which has a fully developed open branch, then we can conclude that ‘¬(φ ↔ ψ)’ is consistent, and hence that ‘(φ ↔ ψ)’ is not logically valid, and hence that φ and ψ are not logically equivalent.

   Examples: use tableaux to prove the two examples above.

6. <u>Exercises</u>

   For each pair of closed wffs below, either prove that they are logically equivalent or prove that they are not:

   a. ‘∃*x*F*x*’ and ‘¬∀*x*¬F*x*’
   b. ‘∀*x*F*x*’ and ‘∀*y*F*y*’
   c. ‘∀*x*∃*y*F*xy*’ and ‘∃*y*∀*x*F*xy*’
   d. ‘∃*x*F*x*’ and ‘∀*x*∃*x*F*x*’
   e. ‘(∀*x*F*x* → A)’ and ‘∃*x*(F*x* → A)’
   f. ‘(A → ∃*x*F*x*)’ and ‘∃*x*(A → F*x*)’
   g. ‘∀*x*((F*x* ∧ G*x*) → H*x*)’ and ‘∀*x*((F*x* ∧ ¬H*x*) → ¬G*x*)’

<u>Entailment</u>

1. If Γ is a set of closed wffs and φ is a closed wff then Γ is said to **entail** φ just in case every interpretation on which every wff in Γ is true is an interpretation on which φ is true. We write ‘Γ ⊨ φ’ to mean that Γ entails φ.

2. We can prove that Γ does *not* entail φ by producing an interpretation on which every wff in Γ is true and φ is false.

   Example: ‘(∃*x*F*x* ∧ E*x*G*x*)’ does not entail ‘∃*x*(F*x* ∧ G*x*)’. *Proof*. Let I be the following interpretation: the domain is {1, 2}; ‘F’ denotes {1}; ‘G’ denotes {2}. Then ‘∃*x*F*x*’ is true on I (because there is something in the set that ‘F’ denotes) and ‘∃*x*G*x*’ is true on I (because there is something in the set that ‘G’ denotes), and so ‘(∃*x*F*x* ∧ E*x*G*x*)’ is true on I. But ‘∃*x*(F*x* ∧ G*x*)’ is false on I (because there is nothing both in the set that ‘F’ denotes and in the set that ‘G’ denotes). So there is an interpretation on which ‘(∃*x*F*x* ∧ E*x*G*x*)’ is true and ‘∃*x*(F*x* ∧ G*x*)’ is false, so ‘(∃*x*F*x* ∧ E*x*G*x*)’ does not entail ‘∃*x*(F*x* ∧ G*x*)’. ■

3. We can prove that Γ *does* entail φ by arguing that there is no interpretation on which every wff in Γ is true and φ is false.

Example: ‘$\exists x(Fx \wedge Gx)$’ entails ‘$(\exists xFx \wedge ExGx)$’. *Proof*. Suppose not. Then there is an interpretation I on which ‘$\exists x(Fx \wedge Gx)$’ is true and ‘$(\exists xFx \wedge ExGx)$’ is false. Since ‘$\exists x(Fx \wedge Gx)$’ is true on I, there is something in the domain of I which is both in the set that ‘F’ denotes and in the set that ‘G’ denotes. So there is something in the domain of I which is in the set that ‘F’ denotes and there is something in the domain of I which is in the set that ‘G’ denotes. So ‘$(\exists xFx \wedge ExGx)$’ is true on I. But ‘$(\exists xFx \wedge ExGx)$’ is false on I. So there is no such I, and ‘$\exists x(Fx \wedge Gx)$’ entails ‘$(\exists xFx \wedge ExGx)$’. ■

4. Result: Γ entails φ iff $\Gamma \cup$ ‘$\neg\varphi$’ is inconsistent. That is, $\Gamma \models \varphi$ iff Γ, ‘$\neg\varphi$’ $\models$.

   Proof. Γ entails φ iff there is no interpretation on which every wff in Γ is true and φ is false. That is, iff there is no interpretation on which every wff in Γ is true and ‘$\neg\varphi$’ is true. That is, iff $\Gamma \cup$ ‘$\neg\varphi$’ is inconsistent. ■

5. So we can use tableau to check whether or not a set of closed wffs Γ entails a closed wff φ: use tableau to check whether or not $\Gamma \cup$ ‘$\neg\varphi$’ is consistent. If we can produce a tableau whose root wffs are the wffs in $\Gamma \cup$ ‘$\neg\varphi$’ and which is closed, then we can conclude that $\Gamma \cup$ ‘$\neg\varphi$’ is inconsistent, and hence that Γ entails φ; if we can produce a tableau whose root wffs are the wffs in $\Gamma \cup$ ‘$\neg\varphi$’ and which has a fully developed open branch, then we can conclude that $\Gamma \cup$ ‘$\neg\varphi$’ is consistent, and hence that Γ does not entail φ.

   Examples: use tableaux to prove the two examples above.

6. <u>Exercises</u>

   Prove or disprove:
   a. ‘$\forall xFx$’ $\models$ ‘$\exists xFx$’
   b. ‘$\forall x(Ax \vee Bx)$’ $\models$ ‘$(\forall xAx \vee \forall xBx)$’
   c. ‘$\forall x\exists yFxy$’ $\models$ ‘$\forall y\exists x\neg Fxy$’
   d. ‘$\forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz)$’ $\models$ ‘$\forall x\forall y\forall z((\neg Rxy \wedge \neg Ryz) \rightarrow \neg Rxz)$’
   e. ‘$\forall x\forall y\forall z((Fxy \wedge Fyz) \rightarrow Fxz)$’, ‘$\forall x\forall y(Fxy \rightarrow Fyx)$’, ‘$\exists x\exists yFxy$’ $\models$ ‘$\forall xFxx$’
   f. ‘$\forall x\exists y\forall z(z = y \leftrightarrow Fxz)$’ $\models$ ‘$\forall x\exists y\forall z(z = y \leftrightarrow Fxy)$’
   g. ‘$\exists x\ x = x$’, ‘$\forall x\exists y(Fx \rightarrow (Fy \wedge \neg x = y))$’ $\models$ ‘$\exists x\exists y((Fx \wedge Fy) \wedge \neg x=y)$’
   h. ‘$\exists x(Gxx \leftrightarrow \exists y\forall zFyz)$’ $\models$ ‘$\forall z\exists yFyz$’

Compactness

1. Result (**Compactness theorem for QC**):

   a. For every set of closed wffs Γ: if Γ is inconsistent then some finite subset of Γ is inconsistent.
   b. For every set of closed wffs Γ: if every finite subset of Γ is consistent then Γ is consistent.
   c. For every closed wff φ and every set of closed wffs Γ: if Γ ⊨ φ then Γ' ⊨ φ for some finite subset Γ' of Γ.

   Proof (of the formulation in (b)). Rather than proving the compactness theorem from first principles, as we did for PC, we will prove it by appealing to the soundness and completeness of the tableau system for QC. The result is trivial if Γ is a finite set, so we need only consider the case in which Γ is an infinite set. So suppose that Γ is an infinite set of closed wffs of QC which is finitely consistent (i.e. every finite subset of Γ is consistent). Since the wffs of QC are denumerable, so are the wffs in Γ. Suppose that we have ordered the wffs in Γ as $<\varphi_1, \varphi_2, \ldots>$. We shall develop in stages a fully developed tableau whose root wffs are the wffs in Γ. Stage 1: start with φ_1 as the only root wff; fully develop the tableau; since $\{\varphi_1\}$ is a finite subset of Γ it is consistent, and hence the tableau will not close at this stage (by the soundness of the tableau system for QC). Stage 2: add φ_2 to the root wffs; continue developing the tableau until it is fully developed; since $\{\varphi_1, \varphi_2\}$ is a finite subset of Γ it is consistent, and hence the tableau will not close at this stage (by the soundness of the tableau system for QC). And so on. In this manner we will get a fully developed tableau whose root wffs are the wffs in Γ, and at no stage will it close, so it will be open. Thus, the wffs in Γ are consistent (by the completeness of the tableau system for QC). ■