<u>PHIL 331/MATH 281: Week 2</u>

<u>The syntax of PC</u>

1. To specify the syntax of PC we specify a set of **symbols**, then a set of **well-formed formulae** (**wffs**) (or: *sentences*).

a. Symbols:

We have two **brackets**: '(' and ')' (or: '[' and ']'). These are just for punctuation, to avoid structural ambiguity.

We have an infinite number of **sentence letters**: 'A', 'B', 'C', 'P_1', 'P_2', 'R', … Note the optional subscripts.

We have five **connectives** (or: *truth functional connectives*, *sentential connectives, truth functors*): '¬' (or: '~'), '∧' (or: '•', '&'), '∨', '→' (or: '⊃', '⇒'), '↔' (or: '≡', '⇔'). These are called **corner**, **hat**, **wedge**, **arrow**, and **double arrow**.

b. Wffs:

A finite string of symbols is called a **formula**. Examples: '))→C', '∧∧∧∧', 'AB∧', '(¬A)', 'A ∨ B'. Note that a formula may include spaces (for readability). The **length** of a formula is the number of **symbol tokens** that it contains.

The class of wffs is defined **recursively** according to the following **formation rules**:

i. Every sentence letter is a wff
ii. If φ is a wff then so is '¬φ'
iii. If φ and ψ are wffs then so are '(φ ∧ ψ)', '(φ ∨ ψ)', '(φ → ψ)', and '(φ ↔ ψ)'
iv. Nothing else is a wff

Examples of wffs:
'A', '¬A_1', '(A_2 ∧ ¬B)', '(A → (¬B ∨ C))', '((¬B ∨ B) → (A ∧ C))', '((¬A ↔ A) ↔ (C → (B ∨ C)))'.

Examples of non-wffs:
'))→C', '∧∧∧∧', 'AB∧', '(¬A)', 'A ∨ B'.

2. 'φ' and 'ψ' are not symbols in the **object language** (i.e. PC). They are symbols in the **metalanguage** (i.e. English) – the language that we use to talk about the object language. They are used to stand for formulae of PC: sometimes as variables, sometimes as schematic letters, sometimes as names.

3. If φ and ψ are formulae then '$(\varphi \wedge \psi)$' is the expression obtained by concatenating '(' with φ with '$\wedge$' with ψ and with ')'. If we use '+' for the concatenation operation, then '$(\varphi \wedge \psi)$' = '(' + φ + '$\wedge$' + ψ + ')'. Spaces can be added for readability.

   Example: if φ is '$(A \vee B)$' and ψ is 'C' then '$(\varphi \wedge \psi)$' is '(' + '$(A \vee B)$' + '$\wedge$' + 'C' + ')', which is '$((A \vee B) \wedge C)$'.

4. The sentence letters and brackets are called the **non-logical symbols** of the language; the others are called the **logical symbols**.

5. The sentence letters can be ordered alphanumerically: 'A', 'A_1', 'A_2', …, 'B', 'B_1', 'B_2', …

6. Although there are infinitely many sentence letters, there are denumerably many. Here is a bijection between the sentence letters and the natural numbers:

A	A_1	A_2	…
B	B_1	B_2	…
C	C_1	C_2	…
…	…	…	…
Z	Z_1	Z_2	…

1	27	53	…
2	28	54	…
3	29	55	…
…	…	…	…
26	52	78	…

   Think of this as a way of counting the sentence letters. It is onto, since everything is counted at least once, and it is 1-1, since nothing is counted twice.

7. If φ is a wff that consists of a sentence letter, then φ is said to be an **atomic wff** (or: *simple wff*); otherwise it is a **complex wff**. So a wff is complex iff it contains at least one connective.

8. Note that there is a lot of **mentioning** rather than **using** of wffs. It is important not to confuse the two. We will mostly be mentioning wffs, rather than using them. (Just like linguists studying a foreign language.)

9. We can prove that '$(A \rightarrow (\neg B \vee C))$' is a wff as follows:

   a. 'B' is a wff, by the first formation rule (FR1)
   b. So '$\neg B$' is a wff, by FR2
   c. 'C' is a wff, by FR1
   d. So '$(\neg B \vee C)$' is a wff, by FR3
   e. 'A' is a wff, by FR1
   f. So '$(A \rightarrow (\neg B \vee C))$' is a wff, by FR3

10. We can prove that '$(A \vee (B \vee C \wedge D))$' is not a wff as follows:

   a. It is not a sentence letter, so it is not a wff by FR1
   b. It is not of the form '$\neg\varphi$', for some wff φ, so is not a wff by FR2

c. It is not of the form ‘$(\varphi \wedge \psi)$’, for some wffs φ and ψ, because ‘D)’ is not a wff (this itself needs proof)
d. It is not of the form ‘$(\varphi \vee \psi)$’, for some wffs φ and ψ, because ‘$(B \vee C \wedge D)$’ is not a wff (this needs proof), and ‘$C \wedge D)$’ is not a wff (this needs proof)
e. It is not of the form ‘$(\varphi \rightarrow \psi)$’
f. It is not of the form ‘$(\varphi \leftrightarrow \psi)$’
g. So, by (c) – (f), it is not a wff by FR3
h. So it is not a wff, by FR4

11. Exercises

a. Which of the following formulae of PC are wffs?
   i. ‘$(\rightarrow AB)$’
   ii. ‘$A \wedge B$’
   iii. ‘$\neg(A \vee B)$’
   iv. ‘$A \rightarrow (A \vee B)$’
   v. ‘$(\neg(A \wedge B))$’
   vi. ‘$\neg(A \rightarrow \neg A)$’
   vii. ‘$(AB\wedge)$’
   viii. ‘$((P) \rightarrow Q)$’
   ix. ‘$(P \wedge (Q \vee P_1))$’
   x. ‘$((P \vee P) \rightarrow P) \wedge Q)$’
   xi. ‘$(((P \leftrightarrow Q) \leftrightarrow P) \leftrightarrow Q)$’
   xii. ‘$(P \leftrightarrow Q \rightarrow Q)$’
   xiii. ‘$(\neg(P \wedge Q) \rightarrow (P \vee Q))$’
   xiv. ‘$\neg(\neg P)$’

b. Prove that the following are wffs:
   i. ‘$(P \wedge ((\neg Q \rightarrow R) \vee S))$’
   ii. ‘$\neg((A \rightarrow \neg B) \rightarrow (\neg A \rightarrow B))$’

c. Prove that the following are not wffs:
   i. ‘$(P \vee Q \vee R)$’
   ii. ‘$(A \rightarrow (A \wedge (\neg(B))))$’

d. Prove that the set of wffs of PC is denumerable (hint: find a way to count them)

Unique decomposition

1. Every complex wff has at least one of the forms ‘$\neg\varphi$’, ‘$(\varphi \wedge \psi)$’, ‘$(\varphi \vee \psi)$’, ‘$(\varphi \rightarrow \psi)$’, or ‘$(\varphi \leftrightarrow \psi)$’ (for otherwise by the last formation rule it would not be a wff).

Note: to say that a wff χ has the form ‘$(\varphi \wedge \psi)$’ is to say that for some wffs φ and ψ, χ = ‘$(\varphi \wedge \psi)$’.

2. Result: Every wff has the same number of left and right brackets.

Proof. We will proceed by **mathematical induction** on the number of connectives in the wff. That is, we will show that for each $n \in \{0, 1, 2, \ldots\}$ any wff with n connectives has the same number of left and right brackets. If φ is a wff with no connectives, then φ is a sentence letter, and φ has the same number of left and right brackets (zero). So the result is true for $n = 0$. Now suppose that the result is true for $n = 0, 1, 2, \ldots k$, for some $k \geq 0$. Suppose that φ is a wff with $k + 1$ connectives. Then φ is of one of the following forms: '¬ψ', '(ψ ∧ ψ')', '(ψ ∨ ψ')', '(ψ → ψ')', or '(ψ ↔ ψ')'. In each case, ψ and ψ' are wffs with k or fewer connectives, and so, by supposition, have the same number of left and right brackets. So in each case, φ has the same number of left and right brackets. So the result is true for $n = k + 1$. Since the result is true for $n = 0$, it also true for $n = 1$. Since the result is true for $n = 0$ and $n = 1$, it is also true for $n = 2$. And so on. So the result is true for all $n \in \{0, 1, 2, \ldots\}$. ■

3. Result: Every proper initial segment of a wff that contains at least one bracket contains more left brackets than right brackets.

Proof. We will again proceed by mathematical induction on the number of connectives in the wff, n. If φ is a wff with no connectives then φ is a sentence letter and has no proper initial segments. So the result is (vacuously) true for $n = 0$. Now suppose that the result is true for $n = 0, 1, 2, \ldots k$, for some $k \geq 0$. Suppose that φ is a wff with $k + 1$ connectives. Then φ has one of the following forms: '¬ψ', '(ψ ∧ ψ')', '(ψ ∨ ψ')', '(ψ → ψ')', or '(ψ ↔ ψ')'. In each case, ψ and ψ' are wffs with k or fewer connectives, and so, by supposition, every proper initial segment of ψ and ψ' that contains at least one bracket contains more left brackets than right brackets. So in each case, every proper initial segment of φ that contains at least one bracket contains more left brackets than right brackets. So the result is true for $n = k + 1$. Since the result is true for $n = 0$, it also true for $n = 1$. Since the result is true for $n = 0$ and $n = 1$, it is also true for $n = 2$. And so on. So the result is true for all $n \in \{0, 1, 2, \ldots\}$. ■

4. Result (**unique decomposition**): (a) If φ = '¬ψ', for some wff ψ, and φ = '¬ψ'', for some wff ψ', then ψ = ψ'; (b) If φ = '¬ψ', for some wff ψ, then φ ≠ '(χ C χ'), for any wffs χ and χ' and 2-place connective C; (c) If φ = '(ψ C χ)', for some wffs ψ and χ and some 2-place connective C, and φ = '(ψ' C' χ')' for some wffs ψ' and χ' and some 2-place connective C', then ψ = ψ' (and therefore C = C' and χ = χ').

Note that this result would not hold if we did not include brackets in the formation rules: if φ is 'A ∨ B ∧ C' then φ = 'ψ C χ', where ψ = 'A', C = '∨' and χ = 'B ∧ C', and also φ = 'ψ' C' χ'', where ψ' = 'A ∨ B', C' = '∧' and χ' = 'C'.

Proof. Only (c) stands in need of proof. Suppose that ψ ≠ ψ'. So one must be shorter than the other, or else they would not be distinct. Assume, without loss of generality, that ψ is shorter than ψ'. So ψ' = 'ψ C …'. Note that ψ is a proper initial segment of ψ', and must contain at least one bracket, because the symbol following ψ is C, a 2-place connective. So by the second result above we have that ψ contains more left brackets than right brackets. But by the first result we have that ψ contains the same number of left and right brackets – a contradiction. So $\psi = \psi'$. ■

5. Because of unique decomposition we can define the **main connective** and the **immediate constituent(s)** of a complex wff:

   a. For ‘$\neg\varphi$’: main connective ‘$\neg$’, immediate constituent φ
   b. For ‘$(\varphi \wedge \psi)$’: main connective ‘$\wedge$’, immediate constituents φ and ψ
   c. For ‘$(\varphi \vee \psi)$’: main connective ‘$\vee$’, immediate constituents φ and ψ
   d. For ‘$(\varphi \rightarrow \psi)$’: main connective ‘$\rightarrow$’, immediate constituents φ and ψ
   e. For ‘$(\varphi \leftrightarrow \psi)$’: main connective ‘$\leftrightarrow$’, immediate constituents φ and ψ

   Note that we could not do this if we did not include brackets in the formation rules: what is the main connective and immediate constituents of ‘$A \vee B \wedge C$’?

6. We sometimes classify complex wffs in the following way:

   a. ‘$\neg\varphi$’ is a **negation**
   b. ‘$(\varphi \wedge \psi)$’ is a **conjunction**; φ and ψ are its **conjuncts**
   c. ‘$(\varphi \vee \psi)$’ is called a **disjunction**; φ and ψ are its **disjuncts**
   d. ‘$(\varphi \rightarrow \psi)$’ is a **conditional**; φ is called its **antecedent**, and ψ is its **consequent**
   e. ‘$(\varphi \leftrightarrow \psi)$’ is a **biconditional**

7. Each wff has a well-defined **constituent structure** which can be represented as a **tree**. There are at least two ways of doing this. Example: ‘$(A \wedge (B \vee C))$’:

8. Note that there is no need to argue by induction over just the natural numbers – any countable set will do (can define a first member and a successor function).

9. <u>Exercises</u>

   a. For each of the following wffs, identify its main connective, classify it as a negation, conjunction, disjunction, conditional, or biconditional, and give its immediate constituents:

i. ‘$((A \wedge A) \wedge \neg C)$’
ii. ‘$((A \wedge B) \to \neg(A \vee B))$’
iii. ‘$\neg\neg(A \vee (A \to (A \wedge A)))$’
iv. ‘$((P \wedge Q) \to (R \vee P))$’
v. ‘$(\neg(R \vee P) \to (P \wedge Q))$’
vi. ‘$(((P \wedge Q) \to R) \vee P)$’
vii. ‘$\neg((R \vee P) \to (P \wedge Q))$’

b. Draw a tree to show the constituent structure of the following wffs:
   i. ‘$((A \wedge A) \wedge \neg C)$’
   ii. ‘$((A \wedge B) \to \neg(A \vee B))$’
   iii. ‘$\neg\neg(A \vee (A \to (A \wedge A)))$’

c. Prove the following results:
   i. Each left bracket corresponds to a unique right bracket, which to occurs to the right of it.
   ii. Each right bracket corresponds to a unique left bracket, which to occurs to the left of it.
   iii. If a bracket occurs between a matched pair of brackets, then its match also occurs between the matched pair.

d. Prove that for any $n > 0$ there is a wff of length n.

e. Suppose that we modify the second formation rule from ‘$\neg\varphi$’ to ‘$(\neg\varphi)$’. Prove that there would be no wffs of length 2, 3, or 6, but there would be wffs of any other positive length.

f. Suppose that φ is a wff. Let c be the number of occurrences of binary connectives in φ, and let s be the number of occurrences of sentence letters (so if φ is ‘$(A \to \neg A)$’ then c is 1 and s is 2). Prove by induction that $s = c + 1$.

g. Suppose that φ is a wff not containing ‘$\neg$’.
   i. Prove that the length of φ is odd.
   ii. Prove that more than a quarter of the symbols are sentence letters. (Hint: show by induction that the length of φ is $4k + 1$, for some $k \geq 0$, and that the number of sentence letters is $k + 1$.)

h. Suppose that instead of left and right brackets we use a single symbol in the formation rules, say ‘|’, so that instead of ‘$(A \vee (B \wedge C))$’ we have ‘$|A \vee |B \wedge C||$’. Do we still get unique decomposition?

i. Prove that it suffices to use just left brackets ‘(’ (i.e. instead of ‘$(A \vee (B \wedge C))$’ we have ‘$(A \vee (B \wedge C$’. (Hint: these expressions have the same number of parentheses as connective symbols.)