PHIL 331/MATH 281: Week 4

Models and consistency

1. If a wff φ is true on an interpretation I then we say that I is a **model** of φ. If φ has a model then we say that φ is **consistent** (or: *satisfiable*). Otherwise we say that φ is **inconsistent** (or: *unsatisfiable*, a *contradiction*), and we write $\varphi \models$. '$\models$' is called a **semantic turnstile**, and '$\varphi \models$' is called a **semantic sequent**.

2. If each wff in a set of wffs Γ (possibly infinite) is true on an interpretation I then we say that I is a model of Γ. If Γ has a model then we say that Γ is consistent (or: satisfiable). Otherwise we say that Γ is inconsistent (or: unsatisfiable), and we write $\Gamma \models$.

3. We can *prove* that a wff or set of wffs is consistent by giving a model of the wff or wffs (and showing that it is a model).

   Example: '$\neg(A \rightarrow B)$' is consistent. *Proof*: If I is an interpretation on which 'A' is true and 'B' is false then I is an interpretation on which '$\neg(A \rightarrow B)$' is true. So '$\neg(A \rightarrow B)$' has a model and is therefore consistent. ■

   Others:
   '$(A \vee (A \rightarrow B))$' is consistent
   {'$(A \rightarrow B)$', '$\neg(A \vee B)$'} is consistent

4. We can prove that a wff or set of wffs is *inconsistent* by arguing that it has no model.

   Example: '$(\neg(A \rightarrow B) \wedge (B \vee \neg A))$' is inconsistent. *Proof*: For it to be true we need both '$\neg(A \rightarrow B)$' and '$(B \vee \neg A)$' to be true. For '$\neg(A \rightarrow B)$' to be true we need '$(A \rightarrow B)$' to be false. For '$(A \rightarrow B)$' to be false we need 'A' to be true and 'B' to be false. But then '$(B \vee \neg A)$' is false. ■

   Others:
   {'$(A \vee B)$', '$(A \rightarrow \neg A)$', '$\neg(B \rightarrow B)$'} is inconsistent

5. By means of truth tables we have an **effective procedure** for deciding whether or not a wff is consistent: construct the truth table for the wff; if there is a row with 'T' in the final column then the wff is consistent, otherwise it is inconsistent.

   Example: Is '$(\neg(A \rightarrow A) \vee \neg A)$' consistent? Check by constructing its truth table:

(	¬	(	A	→	A	)	∨	¬	A	)

6. And we have an effective procedure for deciding whether or not a <u>finite</u> set of wffs is consistent: construct a single truth table for the wffs; if there is a row with 'T' in the final column of each wff then the set is consistent, otherwise it is inconsistent.

   Example: Is {'(A → A)', '¬(A ∨ B)'} consistent?

'A'	'B'	'(A → A)'	'¬(A ∨ B)'

7. Why is this an effective procedure? Because (a) it is mechanical (it could be carried out by a computer), (b) for any wff or set of wffs it is guaranteed to stop after a finite number of steps, and (c) after stopping it is guaranteed to deliver a verdict of either consistent or inconsistent.

8. Note that although it is an effective procedure it may not be possible in practice to carry it out to completion. If there are *n* distinct sentence letters involved then the truth table with have 2^n rows. Suppose *n* is 80 and that we can complete one million rows per second. Then it will take 2^{80} microseconds to complete the table. That is about 38 billion years. The universe is thought to be 15 billion years old.

9. <u>Exercises</u>

   a. Prove that each of the following wffs or sets of wffs is consistent:
      i. '((P ↔ Q) ∧ ((P → Q) ∧ ¬P))'
      ii. {'(A ↔ C)', '¬(A ∧ B)'}
      iii. {'(A ↔ (B ↔ C))', '(B → C)', '((A ∧ ¬B) ∨ (B ∧ ¬A))'}

   b. Prove that each of the following wffs or sets of wffs is inconsistent:
      i. '(¬(A → B) ∧ ¬(B → A))'
      ii. {'(P → Q)', '(¬P ↔ Q)', '(Q → ¬Q)'}
      iii. {'(A ↔ (B ↔ C))', '((A ∨ B) → C)', '((A ∧ ¬B) ∨ (B ∧ ¬A))'}

   c. For each of the following sets of wffs, either prove that it is consistent or prove that it is inconsistent:
      i. {'(A → B)', '(B ↔ C)', '((C ∨ D) ↔ ¬B)'}
      ii. {'¬(¬B ∨ A)', '(A ∨ ¬C)', '(B → ¬C)'}
      iii. {'(D → B)', '(A ∨ ¬B)', '¬(D ∧ A)', 'D'}

   d. Is the empty set consistent?

   e. Suppose that Δ and Γ are sets of wffs (possibly infinite) such that Δ ⊆ Γ. Prove:
      i. If Γ is consistent then Δ is consistent
      ii. If Δ is inconsistent then Γ is inconsistent

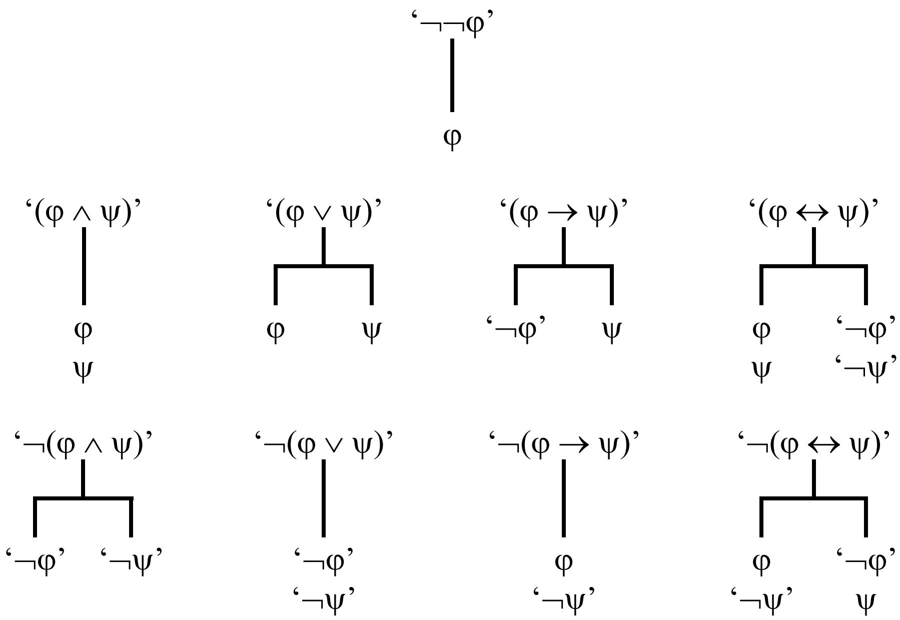
<u>A tableaux system for PC</u>

1. There is another effective procedure for deciding whether or not a wff or finite set of wffs is consistent: the use of **tableaux**.

2. We define a set of tableaux recursively:

   a. Any vertical non-empty but finite list of wffs is a tableau. It has one **branch**.

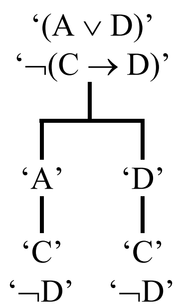
   b. If τ is a tableau, β is a branch of τ, and φ is a wff on β, then the result of applying a **development rule** to φ at the end of the branch is also a tableau. We say that in the resulting tableau φ has been **developed** on each of the resulting branches. The development rules are as follows (each is either **branching** or **non-branching**):



   c. Nothing else is a tableau

3. Every tableau starts with a non-empty list of wffs. These are called its **root wffs**. Every branch of the tableau contains these wffs.

4. Here is a tableau that has been developed from two root wffs:



5. A branch of a tableau is said to be a **fully developed branch** just in case every wff on the branch either is not developable (i.e. is of the form Λ or '¬Λ', for some sentence letter Λ) or has been developed on that branch.

6. A tableau is said to be a **fully developed tableau** just in case each of its branches is fully developed.

7. A branch of a tableau is said to be a **closed branch** just in case in includes a pair wffs of the form φ and '¬φ' (there may be more than one such pair). Otherwise it is said to be an **open branch**.

8. A tableau is said to be a **closed tableau** just in case each of its branches is closed. Otherwise it is said to be an **open tableau**. Every tableau is either open or closed.

9. Result: If a tableau has no open branches then its root wffs are inconsistent.

   Proof. We will first establish the preliminary result that if τ is a tableau with at least one consistent branch (i.e. a branch whose wffs are jointly consistent), and if τ' is a tableau obtained from τ by the application of a development rule to a developable wff φ on a branch β of τ, then τ' also has at least one consistent branch. For suppose that τ has a consistent branch. Suppose that it has a consistent branch that is distinct from β. Since this branch remains unchanged when the development rule is applied to φ on β, it is also a branch of τ'. So τ' has at least one consistent branch. Now suppose that β is the only consistent branch of τ. Since φ is developable it has one the following forms: '¬¬ψ', '(ψ ∧ χ)', '(ψ ∨ χ)', '(ψ → χ)', '(ψ ↔ χ)', '¬(ψ ∧ χ)', '¬(ψ ∨ χ)', '¬(ψ → χ)', or '¬(ψ ↔ χ)'. Suppose that φ = '¬¬ψ', for some wff ψ. Then applying the appropriate development rule to φ on β yields a branch β' = β + ψ (i.e. a branch obtained by appending ψ to the end of β). Since β is consistent there is an interpretation on which all the wffs in β are true. So it is an interpretation on which φ is true. Since φ = '¬¬ψ' is true iff ψ is true, this is an interpretation on which ψ is true as well. So it is an interpretation on which all the wffs in β' = β + ψ are true. So β' is a consistent branch of τ'. So τ' has at least one consistent branch. Similar reasoning shows that if φ has any of the forms '(ψ ∧ χ)', '¬(ψ ∨ χ)', or '¬(ψ → χ)' then applying the appropriate development rule to φ on β yields a branch β' of τ' which is consistent (these are all non-branching rules), so that τ' has at least one consistent branch. Now suppose that φ = '(ψ ∨ χ)', for some wffs ψ and χ. Then applying the appropriate development rule to φ on β yields branches β' = β + ψ and β'' = β + χ. Since β is consistent there is an interpretation on which all the wffs in β are true. So it is an interpretation on which φ is true. Since φ = '(ψ ∨ χ)' is true iff either ψ is true or χ is true, this is an interpretation on which either ψ is true as well or χ is true as well (or both). So it is an interpretation on which either all the wffs in β' = β + ψ are true or all the wffs in β'' = β + χ are true. So either β' or β'' is a consistent branch of τ'. So τ' has at least one consistent branch. Similar reasoning shows that if φ has any of the forms '(ψ → χ)', '(ψ ↔ χ)', '¬(ψ ∧ χ)', or '¬(ψ ↔ χ)' then applying the appropriate development rule to φ on β yields branches β' and β'' of τ' one of which is consistent (these are all branching rules), so that τ' has at least one consistent branch. This establishes the preliminary result. Now suppose that τ is a tableau with no open branches. This means that every branch is closed. This means that every branch is inconsistent. For if a branch is closed then it contains a pair of wffs φ and '¬φ', and there is no interpretation on which these wffs are both true. So τ has no consistent branches. But that means that its root wffs must be inconsistent. For τ has been developed from its root wffs by the application of zero or more development rules. By the preliminary result above if these root wffs were consistent then applying a development rule would always yield a tableau with at least one consistent branch. But τ has no consistent branches. So its root wffs are inconsistent. ■

10. So if we can produce a tableau whose root wffs are the members of Γ and which has no open branches then we can conclude that Γ is inconsistent (the branches need not be fully developed).

    Example. Here is a tableau whose root wffs are '(P ∧ (Q ∨ R))' and '¬((P ∧ Q) ∨ (P ∧ R))' and which has no open branches:

    (P ∧ (Q ∨ R))
    ¬((P ∧ Q) ∨ (P ∧ R))

11. Result: If a tableau has a fully developed open branch then its root wffs are consistent.

    Proof. Suppose that τ is a tableau with a fully developed open branch β. Define an interpretation I of the language as follows: for all sentence letters Λ, if Λ is on β then |Λ| = T, otherwise |Λ| = F. We shall now establish the preliminary result that on this interpretation every wff on β is true. We shall proceed by induction on the number n of connectives in the wffs on β. Suppose that φ is a wff on β with no connectives. Then φ is a sentence letter. By the definition of the interpretation I, |φ| = T. So every wff on β that has no connectives is true on the interpretation I. So the result is true for $n = 0$. Now suppose the result is true for $n = 0, 1, \ldots, k$, for some $k \geq 0$, and that φ is a wff on β with $k + 1$ connectives. Since φ has at least one connective it has one of the following forms: '¬ψ', '(ψ ∧ χ)', '(ψ ∨ χ)', '(ψ → χ)', or '(ψ ↔ χ)'. Suppose that φ = '¬ψ', for some wff ψ. Then φ is of one of the following forms: '¬Λ', '¬¬ψ', '¬(ψ ∧ χ)', '¬(ψ ∨ χ)', '¬(ψ → χ)', or '¬(ψ ↔ χ)'. Suppose that φ = '¬Λ', for some sentence letter Λ. Then '¬Λ' is on β. Since β is an open branch this means that Λ is not on β. So, by the definition of the interpretation I, |Λ| = F. So |φ| = |'¬Λ'| = T. Suppose instead that φ = '¬¬ψ', for some wff ψ. Then φ is a developable wff. Since β is a fully developed branch, φ has been developed on β. So ψ also occurs on β. But since ψ is a wff on β that has fewer connectives than φ, the result is true for ψ. So $|\psi| = T$. So $|\varphi| = |$‘$\neg\neg\psi$’$| = T$. Suppose instead that $\varphi =$ ‘$\neg(\psi \wedge \chi)$’, for some wffs ψ and χ. Then φ is a developable wff. Since β is a fully developed branch, φ has been developed on β. So either ‘$\neg\psi$’ or ‘$\neg\chi$’ also occurs on β. Suppose that ‘$\neg\psi$’ occurs on β. Since ‘$\neg\psi$’ is a wff on β that has fewer connectives than φ, the result is true for ‘$\neg\psi$’. So $|$‘$\neg\psi$’$| = T$. So $|\psi| = F$. So $|\varphi| = |$‘$\neg(\psi \wedge \chi)$’$| = T$. Similarly, if ‘$\neg\chi$’ occurs on β then $|\varphi| = T$. Either way, $|\varphi| = T$. In a similar way we can show that if φ has any of the forms ‘$\neg(\psi \vee \chi)$’, ‘$\neg(\psi \rightarrow \chi)$’, or ‘$\neg(\psi \leftrightarrow \chi)$’ then $|\varphi| = T$. And we can show that if φ has any of the forms ‘$(\psi \wedge \chi)$’, ‘$(\psi \vee \chi)$’, ‘$(\psi \rightarrow \chi)$’, or ‘$(\psi \leftrightarrow \chi)$’ then $|\varphi| = T$. So the result is true for $n = k + 1$. This establishes the preliminary result. We thus have an interpretation on which all the wffs in β are true. But β contains the root wffs of τ. So this is an interpretation on which all of the root wffs of τ are true. So the root wffs of τ are consistent. ■

12. So if we can produce a tableau whose root wffs are the members of Γ and which has a fully developed open branch then we can conclude that Γ is consistent. Moreover, from this open branch we can read off a model for Γ.

Example. Here is a tableau whose root wffs are ‘$((P \wedge Q) \vee R)$’ and ‘$\neg((P \vee Q) \wedge R)$’ and which has a fully developed open branch:

$$((P \wedge Q) \vee R)$$
$$\neg((P \vee Q) \wedge R)$$

13. The tableau method for PC gives us an effective procedure for deciding whether or not a finite set of wffs Γ is consistent: produce a fully developed tableau that starts with the wffs in Γ (every tableau with a finite number of root wffs can be fully developed in a finite number of steps – each time we apply a development rule to a wff the resulting wffs have fewer connectives, and we only need to apply a development rule to a wff once); if the tableau is closed then Γ is inconsistent; if the tableau is open then Γ is consistent (and a model for Γ can be read off any open branch).

If a branch closes along the way then we can stop developing the branch, because even when fully developed it will be closed. If all the branches close along the way then we can stop developing the tableau and conclude that Γ is inconsistent, even if the tableau is not fully developed. But we cannot conclude that Γ is consistent until we get a fully developed open branch.

14. Let Γ be a non-empty but finite set of wffs. Define '$\Gamma \vdash$' to mean that there is a closed tableau whose root wffs are the members of Γ. '$\vdash$' is called a **syntactic turnstile**, and '$\Gamma \vdash$' is called a **syntactic sequent**. We have proved the following results:

Soundness: If $\Gamma \vdash$ then $\Gamma \models$
Completeness: If $\Gamma \models$ then $\Gamma \vdash$

15. If $\Gamma \vdash$ say that Γ is **syntactically inconsistent**, and if $\Gamma \models$ say that Γ is **semantically inconsistent**. Then we have proved that Γ is syntactically inconsistent iff it is semantically inconsistent: $\Gamma \vdash$ iff $\Gamma \models$.

16. <u>Exercises</u>

a. Prove that the following wffs or sets of wffs are inconsistent by constructing an appropriate tableau:
   i. '$(\neg(A \to B) \wedge (B \vee \neg A))$'
   ii. {'$(P \vee Q)$', '$\neg((P \to Q) \to Q)$'}
   iii. {'$((P \to Q) \to Q)$', '$\neg(P \vee Q)$'}
   iv. {'$(P \wedge (Q \vee R))$', '$\neg((P \wedge Q) \vee (P \wedge R))$'}
   v. {'$((P \vee Q) \to (P \wedge Q))$', '$(\neg P \leftrightarrow Q)$'}
   vi. {'$(P \to Q)$', '$(\neg P \leftrightarrow Q)$', '$(Q \to \neg Q)$'}
   vii. {'$(A \leftrightarrow (B \leftrightarrow C))$', '$((A \vee B) \to C)$', '$((A \wedge \neg B) \vee (B \wedge \neg A))$'}
   viii. {'$(P \to Q)$', '$(Q \to R)$', '$(R \to S)$', '$\neg(P \to S)$'}

b. Prove that the following wffs or sets of wffs are consistent by constructing an appropriate tableau, and give a model for each:
   i. '$\neg(A \to B)$'
   ii. '$(A \vee (A \to B))$'
   iii. '$(\neg(A \to A) \vee \neg A)$'
   iv. '$((P \leftrightarrow Q) \wedge ((P \to Q) \wedge \neg P))$'
   v. {'$(A \to B)$', '$\neg(A \vee B)$'}
   vi. {'$((\neg P \to Q) \to P)$', '$\neg P$'}
   vii. {'$(P \vee R)$', '$\neg((P \to Q) \to P)$'}
   viii. {'$(A \leftrightarrow (B \leftrightarrow C))$', '$(B \to C)$', '$((A \wedge \neg B) \vee (B \wedge \neg A))$'}

c. For each of the following wffs or sets of wffs, decide whether or not it is consistent by constructing an appropriate tableau:
   i. '$(\neg(A \to B) \wedge \neg(B \to A))$'
   ii. {'$(A \to B)$', '$(B \leftrightarrow C)$', '$((C \vee D) \leftrightarrow \neg B)$'}
   iii. {'$((P \vee Q) \wedge (P \wedge R))$', '$\neg(P \vee (Q \wedge R))$'}
   iv. {'$(A \vee B)$', '$(A \to \neg A)$', '$\neg(B \to B)$'}
   v. {'$\neg(\neg B \vee A)$', '$(A \vee \neg C)$', '$(B \to \neg C)$'}
   vi. {'$(D \to B)$', '$(A \vee \neg B)$', '$\neg(D \wedge A)$', 'D'}
   vii. {'$(P \vee (Q \wedge R))$', '$\neg((P \vee Q) \wedge (P \wedge R))$'}
   viii. {'$(P \leftrightarrow Q)$', '$\neg((P \vee Q) \to (P \wedge Q))$'}