PHIL 331/MATH 281: Week 5

Tautologies

1. If a wff φ is true on every interpretation then we say that φ is a **tautology** (or: *valid*), and we write $\models \varphi$.

   Examples:
   '$(\varphi \vee \neg\varphi)$' is a tautology, for any wff φ. This is the **law of excluded middle**.
   '$\neg(\varphi \wedge \neg\varphi)$' is a tautology, for any wff φ. This is the **law of non-contradiction**.

2. We can prove that a wff is *not* a tautology by giving an interpretation on which it is false.

   Example: '$(A \vee (\neg A \rightarrow A))$' is not a tautology. *Proof*: Let I be an interpretation on which 'A' is false. Then '$(A \vee (\neg A \rightarrow A))$' is false on this interpretation. So there is an interpretation on which '$(A \vee (\neg A \rightarrow A))$' is false. So it is not a tautology.■

3. We can prove that a wff *is* a tautology by arguing that there is no interpretation on which it is false.

   Example: '$((A \rightarrow B) \vee (B \rightarrow A))$' is a tautology. *Proof*: Suppose that there is an interpretation I on which it is false. Then '$(A \rightarrow B)$' and '$(B \rightarrow A)$' are both false on I. But if '$(A \rightarrow B)$' is false on I then 'A' is true on I, and if '$(B \rightarrow A)$' is false on I then 'A' is false on I. So 'A' is both true and false on I. But there is no such I. So there is no interpretation on which it is false. So it is a tautology. ■

4. By means of truth tables we have an effective procedure for determining whether or not a wff is a tautology: construct its truth table; it is a tautology just in case its truth table has all 'T's in the final column.

   Examples:

(A	∨	(¬	A	→	A))

((A	→	B)	∨	(B	→	A))

5. Result: φ is a tautology iff '$\neg\varphi$' is inconsistent. (That is, $\models \varphi$ iff '$\neg\varphi$' $\models$.)

   Proof. φ is a tautology iff φ is true on every interpretation. That is, iff '$\neg\varphi$' is false on every interpretation. That is, iff '$\neg\varphi$' is inconsistent. ■

6. This result means that we have another effective procedure for deciding whether or not φ is a tautology: use tableaux to test whether or not '$\neg\varphi$' is consistent.

Examples: '$(A \vee (\neg A \to A))$' and '$((A \to B) \vee (B \to A))$'

'$\neg(A \vee (\neg A \to A))$'

'$\neg((A \to B) \vee (B \to A))$'

7. We thus have three ways of proving that a wff is or is not a tautology: the 'direct' method, the use of truth tables, and the use of tableaux.

8. <u>Exercises</u>

a. For each of the following wffs either prove that it is a tautology or prove that it is not (use any of the three methods):

i. '$(A \wedge \neg(A \vee B))$'
ii. '$(\neg A \to (A \wedge B))$'
iii. '$(A \to (B \to (B \to A)))$'
iv. '$(((A \to B) \to B) \to B)$'
v. '$((B \leftrightarrow (B \to A)) \to A)$'
vi. '$((A \leftrightarrow B) \leftrightarrow (A \leftrightarrow (B \leftrightarrow A)))$'
vii. '$((A \leftrightarrow (\neg B \vee C)) \to (\neg A \to B))$'
viii. '$((A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C))$'
ix. '$(((B \to C) \to (A \to B)) \to (A \to B))$'
x. '$((A \vee (\neg(B \wedge C))) \to ((A \leftrightarrow C) \vee B))$'

b. Define an infinite set of wffs $\{\varphi_0, \varphi_1, \ldots\}$ recursively as follows: φ_0 = 'P', φ_{n+1} = '$(\varphi_n \to P)$'. Prove that φ_n is a tautology iff *n* is odd.

c. Prove:

i. If φ and '$(\varphi \to \psi)$' are tautologies then so is ψ.
ii. If φ is a tautology then any substitution instance of φ is a tautology.
iii. If φ is not a tautology then φ has a substitution instance which is inconsistent.
iv. If φ is consistent then it has a substitution instance which is a tautology.
v. Any wff containing no sentence letter other than 'P' and no connective other than '$\leftrightarrow$' is either a tautology or logically equivalent to 'P'.
vi. Any wff containing no connective other than '$\leftrightarrow$' is either a tautology or equivalent to a formula in which no sentence letter occurs more than once.

Equivalence

1. If wffs φ and ψ have the same truth value on an interpretation I then we say that φ and ψ are **materially equivalent** on I.

   This is relative to an interpretation: φ and ψ might be materially equivalent on one interpretation but not on another. Example: 'A' and 'B'.

2. If wffs φ and ψ have the same truth value on every interpretation then we say that φ and ψ are **logically equivalent**, and we write $\varphi \mathrel{=\!|\!\models} \psi$. This is not relative to an assignment.

   Examples:
   'A' and '$\neg\neg$A' are logically equivalent
   '(A $\rightarrow$ B)' and '($\neg$A $\vee$ B)' are logically equivalent

3. We can prove that two wffs are *not* logically equivalent by giving an interpretation on which they have different truth values.

   Example. '(A $\wedge$ $\neg$B)' and '(A $\wedge$ B)' are not logically equivalent. *Proof*. Let I be an interpretation on which 'A' is ___ and 'B' is ___. Then I is an interpretation on which '(A $\wedge$ $\neg$B)' is ___ and '(A $\wedge$ B)' is ___. So there is an interpretation on which they have different truth values. So they are not logically equivalent. ■

   Another example: '(A $\vee$ B)' and '((A $\rightarrow$ B) $\rightarrow$ A)' are not logically equivalent.

4. We can prove that two wffs *are* logically equivalent by arguing that there is no interpretation on which they have different truth values.

   Example. '(A $\rightarrow$ B)' and '($\neg$A $\vee$ B)' are logically equivalent. *Proof*. Suppose that I is an interpretation on which '(A $\rightarrow$ B)' is false. Then this is an interpretation on which 'A' is true and 'B' is false, and thus an interpretation on which '($\neg$A $\vee$ B)' is false. Suppose that I is an interpretation on which '($\neg$A $\vee$ B)' is false. Then this is an interpretation on which 'A' is true and 'B' is false and thus an interpretation on which '(A $\rightarrow$ B)' is false. So there is no interpretation on which one is false and the other true. So they are logically equivalent. ■

   Another example: '(A $\rightarrow$ (A $\rightarrow$ B))' and '((A $\vee$ A) $\rightarrow$ B)' are logically equivalent.

5. By means of truth tables we have an effective procedure for determining whether or not two wffs are logically equivalent: construct a single truth table for the two wffs; if there is a row in which the wffs have different truth values then they are not logically equivalent; otherwise they are.

Example:

(A	$\vee$	B)	((A	$\rightarrow$	B)	$\rightarrow$	A)

6. Note that ‘$(A \wedge B)$’ and ‘$(C \wedge D)$’ are not logically equivalent.

7. Result: φ and ψ are logically equivalent iff ‘$(\varphi \leftrightarrow \psi)$’ is a tautology.

   Proof: φ and ψ are logically equivalent iff φ and ψ have the same truth value on every interpretation. That is, iff ‘$(\varphi \leftrightarrow \psi)$’ is true on every interpretation. That is, iff ‘$(\varphi \leftrightarrow \psi)$’ is a tautology. ■

8. This result means that we have another effective procedure for deciding whether or not two wffs φ and ψ are logically equivalent: use tableaux to test whether or not ‘$(\varphi \leftrightarrow \psi)$’ is a tautology (i.e. whether or not ‘$\neg(\varphi \leftrightarrow \psi)$’ is consistent).

   Example: ‘$(A \vee B)$’ and ‘$((A \rightarrow B) \rightarrow A)$’:

$$‘\neg((A \vee B) \leftrightarrow ((A \rightarrow B) \rightarrow A))’$$

9. We thus have three ways of proving that two wffs are or are not logically equivalent: the ‘direct’ method, the use of truth tables, and the use of tableaux.

10. Here are some results about logical equivalence:

    Distributivity
    ‘$\neg(\varphi \wedge \psi)$’ is logically equivalent to ‘$(\neg\varphi \vee \neg\psi)$’
    ‘$\neg(\varphi \vee \psi)$’ is logically equivalent to ‘$(\neg\varphi \wedge \neg\psi)$’
    ‘$\neg(\varphi \rightarrow \psi)$’ is logically equivalent to ‘$(\varphi \wedge \neg\psi)$’
    ‘$\neg(\varphi \leftrightarrow \psi)$’ is logically equivalent to ‘$((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi))$’
    ‘$(\varphi \wedge (\psi \vee \chi))$’ is logically equivalent to ‘$((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$’
    ‘$(\varphi \vee (\psi \wedge \chi))$’ is logically equivalent to ‘$((\varphi \vee \psi) \wedge (\varphi \vee \chi))$’
    [the first two are called **De Morgan’s laws**]

Commutativity
'$(\varphi \wedge \psi)$' is logically equivalent to '$(\psi \wedge \varphi)$'
'$(\varphi \vee \psi)$' is logically equivalent to '$(\psi \vee \varphi)$'
'$(\varphi \leftrightarrow \psi)$' is logically equivalent to '$(\psi \leftrightarrow \varphi)$'
[note that '$\rightarrow$' is not commutative]

Associativity
'$(\varphi \wedge (\psi \wedge \chi))$' is logically equivalent to '$((\varphi \wedge \psi) \wedge \chi)$'
'$(\varphi \vee (\psi \vee \chi))$' is logically equivalent to '$((\varphi \vee \psi) \vee \chi)$'
'$(\varphi \leftrightarrow (\psi \leftrightarrow \chi))$' is logically equivalent to '$((\varphi \leftrightarrow \psi) \leftrightarrow \chi)$'
[note that '$\rightarrow$' is not associative]

Interdefinability
'$(\varphi \wedge \psi)$' is logically equivalent to '$\neg(\varphi \rightarrow \neg\psi)$'
'$(\varphi \vee \psi)$' is logically equivalent to '$(\neg\varphi \rightarrow \psi)$'
'$(\varphi \leftrightarrow \psi)$' is logically equivalent to '$\neg((\varphi \rightarrow \psi) \rightarrow \neg(\psi \rightarrow \varphi))$'
'$(\varphi \rightarrow \psi)$' is logically equivalent to '$(\neg\varphi \vee \psi)$'
'$(\varphi \leftrightarrow \psi)$' is logically equivalent to '$((\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi))$'

Other
'$\neg\neg\varphi$' is logically equivalent to φ
'$(\varphi \rightarrow \psi)$' is logically equivalent to '$(\neg\psi \rightarrow \neg\varphi)$' (**Contraposition**)
'$((\varphi \wedge \psi) \rightarrow \chi))$' is logically equivalent to '$(\varphi \rightarrow (\psi \rightarrow \chi))$' (**Exportation**)

11. Result: If φ is a wff that contains occurrences of a wff ψ, and if ψ' is a wff that is logically equivalent to ψ, then replacing one or more occurrences of ψ in φ by ψ' yields a wff that is logically equivalent to φ.

Example:
'$((A \vee B) \rightarrow B)$' is logically equivalent to '$((A \vee \neg\neg B) \rightarrow B)$', '$((A \vee B) \rightarrow \neg\neg B)$', and '$((A \vee \neg\neg B) \rightarrow \neg\neg B)$'.

Proof. Suppose that ψ and ψ' are logically equivalent. Suppose that I is an interpretation. Since ψ and ψ' are logically equivalent, ψ and ψ' have the same truth value on I. So replacing occurrences of ψ in a wff φ by occurrences of ψ' yields a wff φ' with the same truth value as φ on I. This is true for every I, so φ and φ' are logically equivalent. ■

12. Exercises

a. Prove or disprove (use any of the three methods):
   i. '$((A \wedge B) \vee \neg B)$' is logically equivalent to '$(A \vee \neg B)$'
   ii. '$((A \vee B) \wedge \neg B)$' is logically equivalent to '$(A \wedge \neg B)$'
   iii. '$(A \vee (A \wedge B))$' is logically equivalent to 'A'
   iv. '$(A \wedge (A \vee B))$' is logically equivalent to 'A'
   v. '$(A \leftrightarrow B)$' is logically equivalent to '$((A \rightarrow B) \wedge (B \rightarrow A))$'

vi. ‘$(\neg A \vee B)$’ is logically equivalent to ‘$(\neg B \vee A)$’
vii. ‘$\neg(A \leftrightarrow B)$’ is logically equivalent to ‘$(A \leftrightarrow \neg B)$’

b. Prove:
   i. If φ is a tautology then ‘$(\varphi \wedge \psi)$’ is logically equivalent to ψ and ‘$(\varphi \vee \psi)$’ is logically equivalent to φ
   ii. If φ is inconsistent then ‘$(\varphi \wedge \psi)$’ is logically equivalent to φ and ‘$(\varphi \vee \psi)$’ is logically equivalent to ψ
   iii. Any wff containing no connectives other than ‘$\leftrightarrow$’ and ‘$\neg$’ is logically equivalent to a wff containing no connectives other than ‘$\leftrightarrow$’ together with at most one occurrence of ‘$\neg$’.

Entailment

1. If Γ is a set of wffs (possibly empty, possibly infinite) and φ is a wff such that every interpretation on which every wff in Γ is true is an interpretation on which φ is true, then we say that Γ **entails** φ (or: φ *follows from* Γ, φ *is a consequence of* Γ), and we write $\Gamma \models \varphi$.

   Equivalently, $\Gamma \models \varphi$ iff there is no interpretation on which the wffs in Γ are all true and φ is false.

2. Instead of ‘$\{\varphi_1, \ldots, \varphi_n\} \models \varphi$’ we write ‘$\varphi_1, \ldots, \varphi_n \models \varphi$’, and instead of ‘$\{\} \models \varphi$’ we write ‘$\models \varphi$’.

3. Note:
   a. $\models \varphi$ just in case there is no interpretation on which φ is false; that is, just in case φ is a tautology.
   b. We also define ‘$\Gamma \models$’ to mean that there is no interpretation on which every wff in Γ is true. So $\Gamma \models$ just in case Γ is inconsistent.
   c. φ and ψ are logically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$ (i.e. iff $\varphi \models\!\dashv \psi$). (One of the exercises is to prove this.)

4. We can prove that Γ does *not* entail φ by producing an interpretation on which every wff in Γ is true and φ is false.

   Example: We do not have ‘$(A \rightarrow B)$’, ‘$(B \rightarrow A)$’ $\models$ ‘$(A \wedge B)$’. *Proof.* If I is an interpretation on which ‘A’ is __ and ‘B’ is __ then I is an interpretation on which the wffs on the left are all true but the wff on the right is false. ■

5. We can prove that Γ *does* entail φ by arguing that there is no interpretation on which every wff in Γ is true and φ is false.

   Example: We have ‘A’, ‘$(A \rightarrow B)$’ $\models$ ‘$(A \vee B)$’. *Proof.* Suppose there is an interpretation I on which ‘$(A \vee B)$’ is false. Then this is an interpretation on which ‘A’ is false’. So there is no interpretation on which 'A' and '$(A \rightarrow B)$' are both true and '$(A \vee B)$' is false. ■

6. By means of truth tables we have an effective procedure for deciding whether or not $\Gamma \models \varphi$: create a single truth table for the wffs in Γ and φ; $\Gamma \models \varphi$ just in case there is no row in which the wffs in Γ are all true but φ is false.

Example. Here is a truth table that shows that 'A', '$(A \rightarrow B)$' $\models$ '$(A \vee B)$':

A	(A	→	B)	(A	∨	B)

Example. Here is a truth table that shows that we do not have: '$(A \rightarrow B)$', '$(B \rightarrow A)$' $\models$ '$(A \wedge B)$':

(A	→	B)	(B	→	A)	(A	∧	B)

7. Result: Γ entails φ iff $\Gamma \cup \{`\neg\varphi'\}$ is inconsistent. (That is, $\Gamma \models \varphi$ iff Γ,'$\neg\varphi$' $\models$.)

Proof. Γ entails φ iff there is no interpretation on which the wffs in Γ are all true and φ is false. That is, iff there is no interpretation on which the wffs in Γ are all true and '$\neg\varphi$' is true. That is, iff $\Gamma \cup \{`\neg\varphi'\}$ is inconsistent. ■

8. This result means that we have another effective procedure for deciding whether or not Γ entails φ: use tableaux to test whether or not $\Gamma \cup \{`\neg\varphi'\}$ is consistent.

Example: '$(A \rightarrow B)$', '$(B \rightarrow A)$' $\models$ '$(A \wedge B)$'

'$(A \rightarrow B)$'
'$(B \rightarrow A)$'
'$\neg(A \wedge B)$'

9. We thus have three ways of proving that Γ does or does not entail φ: the 'direct' method, the use of truth tables, and the use of tableaux.

10. Exercises

a. Prove or disprove:
   i. '$\neg(P \rightarrow P)$' $\models$ '$(P \rightarrow \neg P)$'
   ii. '$(A \leftrightarrow (B \leftrightarrow C))$' $\models$ '$((A \wedge (B \wedge C)) \vee (\neg A \wedge (\neg B \wedge \neg C)))$'
   iii. '$(A \wedge B)$', '$\neg B$' $\models$ '$\neg A$'
   iv. '$(A \rightarrow B)$', '$(C \wedge \neg B)$' $\models$ '$\neg A$'
   v. '$(A \vee B)$', '$(A \rightarrow C)$', '$(B \rightarrow C)$' $\models$ 'C'
   vi. '$(P \rightarrow (Q \vee R))$', '$(R \rightarrow (P \rightarrow S))$', '$\neg(S \wedge P)$' $\models$ '$(P \rightarrow Q)$'
   vii. $\models$ '$((P \rightarrow Q) \vee (P \rightarrow \neg Q))$'
   viii. '$((P \leftrightarrow Q) \wedge (P \leftrightarrow \neg Q))$' $\models$

b. Prove:
   i. If φ is a tautology then $\Gamma \models \varphi$, for any Γ
   ii. If Γ is inconsistent then $\Gamma \models \varphi$, for any φ
   iii. φ and ψ are logically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$