PHIL 331/MATH 281: Week 6

The greek alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Y	upsilon
ε	E	epsilon	ν	N	nu	φ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	ο	O	omicron	ψ	Ψ	psi
θ	Θ	theta	π	Π	pi	ω	Ω	omega

Semantic entailment generalized

1. If Γ and Δ are sets of wffs (possibly empty, possibly infinite), say that **Γ (semantically) entails Δ**, written $\Gamma \models \Delta$, iff every interpretation on which every wff in Γ is true is an interpretation on which some wff in Δ is true.

2. There are other ways that we can formulate this, because of the following equivalences:

   a. Every F is G
   b. It is not the case that some F is not G
   c. It is not the case that there is an F which is not G
   d. There is no F which is not G

3. From this general definition of entailment we get the previously-discussed special cases:

   a. $\Gamma \models \varphi$ iff every interpretation on which every wff in Γ is true is an interpretation on which φ is true. That is, iff there is no interpretation on which every wff in Γ is true and φ is false.

   b. $\psi \models \varphi$ iff every interpretation on which ψ is true is an interpretation on which φ is true. That is, iff there is no interpretation on which ψ is true and φ is false.

   c. $\psi \;=\!\!|\!\models \varphi$ (i.e. $\psi \models \varphi$ and $\varphi \models \psi$) iff there is no interpretation on which ψ is true and φ is false, and there is no interpretation on which φ is true and ψ is false. That is, iff **ψ and φ are logically equivalent**.

   d. $\Gamma \models$ iff every interpretation on which every wff in Γ is true is an interpretation on which some wff in {} is true. That is, iff there is no interpretation on which every wff in Γ is true and there is no wff in {} that is true. That is, iff there is no interpretation on which every wff in Γ is true. That is, iff **Γ is inconsistent**.

e. $\models \varphi$ iff every interpretation on which every wff in {} is true is an interpretation on which φ is true. That is, iff every interpretation on which there is no wff in {} that is false is an interpretation on which φ is true. That is, iff every interpretation is an interpretation on which φ is true. That is, iff **φ is a tautology**.

f. $\models \Delta$ iff every interpretation on which every wff in {} is true is an interpretation on which some wff in Δ is true. That is, iff every interpretation on which there is no wff in {} that is false is an interpretation on which some wff in Δ is true. That is, iff every interpretation is an interpretation on which some wff in Δ is true. That is, iff **Δ is a tautology**, if we say that a set of wffs Δ is a tautology just in case on every interpretation there is at least one wff in Δ which is true.

g. $\models$ iff every interpretation on which every wff in {} is true is an interpretation on which some wff in {} is true. That is, iff every interpretation on which there is no wff in {} that is false is an interpretation on which some wff in {} is true. But that is **false**.

4. Exercises

a. Suppose that $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ for some $n \geq 1$ and some wffs $\varphi_1, \ldots, \varphi_n$. Let $\Gamma^{\wedge}$ = '$(\varphi_1 \wedge (\varphi_2 \wedge (\ldots (\varphi_{n-1} \wedge \varphi_n) \ldots)$' and $\Gamma^{\vee}$ = '$(\varphi_1 \vee (\varphi_2 \vee (\ldots (\varphi_{n-1} \vee \varphi_n) \ldots)$'. Prove:
   i. $\Gamma \models$ iff $\Gamma^{\wedge}$ is inconsistent
   ii. $\models \Gamma$ iff $\Gamma^{\vee}$ is a tautology

b. If Δ is a set of wffs define '$\neg\Delta$' to be the set of wffs {'$\neg\varphi$': $\varphi \in \Delta$}. Prove that $\models \Delta$ iff '$\neg\Delta$' $\models$.

Some properties of entailment

1. Assumptions
   a. $\varphi \models \varphi$
   b. If $\varphi \in \Gamma$ then $\Gamma \models \varphi$

2. Thinning
   a. If $\Gamma \models \varphi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \models \varphi$
   b. If $\Gamma \models$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \models$
   c. If $\Gamma \models \Delta$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \models \Delta$ (thinning on the left)
   d. If $\Gamma \models \Delta$ and $\Delta \subseteq \Delta'$ then $\Gamma \models \Delta'$ (thinning on the right)

3. Cut (transitivity)
   a. If $\varphi \models \psi$ and $\psi \models \chi$ then $\varphi \models \chi$
   b. If $\Gamma \models \varphi$ and $\Delta, \varphi \models \psi$ then $\Gamma, \Delta \models \psi$

4. Negation
   a. $\Gamma \models \varphi$ iff Γ, '$\neg\varphi$' $\models$

b. φ, ‘$\neg\varphi$’ $\models$
c. If $\Gamma, \varphi \models \psi$ and Γ, ‘$\neg\varphi$’ $\models \psi$ then $\Gamma \models \psi$

5. Conjunction
   a. $\Gamma \models$ ‘$(\varphi \wedge \psi)$’ iff $\Gamma \models \varphi$ and $\Gamma \models \psi$
   b. $\varphi, \psi \models$ ‘$(\varphi \wedge \psi)$’
   c. ‘$(\varphi \wedge \psi)$’ $\models \varphi$

6. Disjunction
   a. Γ, ‘$(\varphi \vee \psi)$’ $\models$ iff $\Gamma \models$ ‘$\neg\varphi$’ and $\Gamma \models$ ‘$\neg\psi$’
   b. If $\Gamma \models \varphi$ then $\Gamma \models$ ‘$(\varphi \vee \psi)$’

7. Conditional
   a. $\Gamma, \varphi \models \psi$ iff $\Gamma \models$ ‘$(\varphi \rightarrow \psi)$’

8. Interchange of equivalents
   a. ‘$(\varphi \leftrightarrow \psi)$’ $\models$ ‘$(\chi \leftrightarrow \chi(\psi/\varphi))$’
   b. If $\models$ ‘$(\varphi \leftrightarrow \psi)$’ then $\models$ ‘$(\chi \leftrightarrow \chi(\psi/\varphi))$’
   c. If $\varphi \;=\!\!\mid\;\models \psi$ then $\chi \;=\!\!\mid\;\models \chi(\psi/\varphi)$

9. Idempotence
   a. ‘$(\varphi \wedge \varphi)$’ $=\!\!\mid\;\models \varphi$
   b. ‘$(\varphi \vee \varphi)$’ $=\!\!\mid\;\models \varphi$

10. Elaboration
    a. $\varphi \;=\!\!\mid\;\models$ ‘$((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi))$’
    b. $\varphi \;=\!\!\mid\;\models$ ‘$((\varphi \vee \psi) \wedge (\varphi \vee \neg\psi))$’

11. Result (uniform substitution): If $\Gamma \models \varphi$ then $\Gamma(\psi/L) \models \varphi(\psi/L)$.

    Example:
    Since ‘$(A \wedge (A \rightarrow B))$’ $\models$ ‘B’ we have that ‘$(A \wedge (A \rightarrow (A \vee B)))$’ $\models$ ‘$(A \vee B)$’

    Proof. Suppose that $\Gamma \models \varphi$. Suppose that it is not the case that $\Gamma(\psi/L) \models \varphi(\psi/L)$. So there is an interpretation on which $\Gamma(\psi/L)$ is true but $\varphi(\psi/L)$ is false. Since the sentence letter L does not occur in $\Gamma(\psi/L)$ or $\varphi(\psi/L)$, we can modify this interpretation, if need be, so that L is assigned whatever the truth value of ψ is, without affecting the truth values of $\Gamma(\psi/L)$ or $\varphi(\psi/L)$. Then we have an interpretation on which Γ is true and φ is false, contrary to the assumption that $\Gamma \models \varphi$. ■

12. <u>Exercises</u>

    a. Prove:
       i. If $\Gamma \models \Delta$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \models \Delta$
       ii. If $\Gamma \models \Delta$ and $\Delta \subseteq \Delta'$ then $\Gamma \models \Delta'$

iii. If $\Gamma \models \varphi$ and $\Delta, \varphi \models \psi$ then $\Gamma, \Delta \models \psi$
iv. If $\Gamma, \varphi \models \psi$ and Γ, '$\neg\varphi$' $\models \psi$ then $\Gamma \models \psi$
v. $\Gamma \models$ '$(\varphi \wedge \psi)$' iff $\Gamma \models \varphi$ and $\Gamma \models \psi$

b. Prove or disprove:
i. $\Gamma \models$ '$\neg\varphi$' if not $\Gamma \models \varphi$
ii. $\Gamma \models$ '$\neg\varphi$' only if not $\Gamma \models \varphi$
iii. $\Gamma \models$ '$(\varphi \vee \psi)$' if either $\Gamma \models \varphi$ or $\Gamma \models \psi$
iv. $\Gamma \models$ '$(\varphi \vee \psi)$' only if either $\Gamma \models \varphi$ or $\Gamma \models \psi$

c. Prove that the following are equivalent:
i. Γ is consistent
ii. There is no wff φ such that both $\Gamma \models \varphi$ and $\Gamma \models$ '$\neg\varphi$'
iii. There is some wff φ such that it is not the case that $\Gamma \models \varphi$

d. Prove that if $\varphi \models \psi$ but φ and ψ have no sentence letters in common then either φ is inconsistent or ψ is a tautology.

e. Prove that if '$\Gamma \models \varphi$' is an incorrect sequent then it has a substitution instance '$\Delta \models \psi$' (also incorrect) such that every member of Δ is a tautology and ψ is a contradiction.

Compactness

1. Result (**Compactness theorem**). Suppose that Γ is a set of wffs (possibly empty, possibly infinite). The we have each of the following equivalent results:

a. If $\Gamma \models \varphi$ then $\Gamma' \models \varphi$ for some finite subset Γ' of Γ.
b. If Γ is inconsistent then some finite subset of Γ is inconsistent.
c. If every finite subset of Γ is consistent then Γ is consistent.

(One of the exercises is to prove that these are equivalent.)

2. Note that the compactness theorem is trivial if Γ is finite (which includes the case in which Γ is empty).

3. Note also that the converse of the compactness theorem is trivial: if $\Gamma' \models \varphi$ for some finite subset Γ' of Γ then $\Gamma \models \varphi$ (this follows by thinning on the left).

4. Proof of the compactness theorem:

Say that Γ is *finitely consistent* just in case every finite subset of Γ is consistent. We shall first establish that if Γ is finitely consistent, then for any wff φ at least one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{$'$\neg\varphi$'$\}$ is finitely consistent. For suppose not. Then neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{$'$\neg\varphi$'$\}$ are finitely consistent. So there is a finite subset Γ_1 of $\Gamma \cup \{\varphi\}$ which is inconsistent, and a finite subset Γ_2 of $\Gamma \cup \{$'$\neg\varphi$'$\}$ which is inconsistent. Since Γ_1 is finite but inconsistent, and since by supposition every finite subset of Γ is consistent, Γ_1 cannot be a subset of Γ. So we must have $\Gamma_1 = \Gamma' \cup \{\varphi\}$, for some finite subset Γ' of Γ. Similarly, since Γ_2 is finite but inconsistent, and since by supposition every finite subset of Γ is consistent, Γ_2 cannot be a subset of Γ. So we must have $\Gamma_2 = \Gamma'' \cup \{\text{‘}\neg\varphi\text{’}\}$, for some finite subset Γ'' of Γ. But then we have that $\Gamma' \cup \Gamma''$ is inconsistent. For suppose that it were consistent. Then there would be an interpretation on which every wff in $\Gamma' \cup \Gamma''$ is true. That is, an interpretation on which every wff in Γ' is true and on which every wff in Γ'' is true. But since $\Gamma_1 = \Gamma' \cup \{\varphi\}$ is inconsistent, this would be an interpretation on which φ is false. And since $\Gamma_2 = \Gamma'' \cup \{\text{‘}\neg\varphi\text{’}\}$ is inconsistent, this would be an interpretation on which ‘$\neg\varphi$’ is false. So it would be an interpretation on which both φ and ‘$\neg\varphi$’ are false. But there is no such interpretation. So $\Gamma' \cup \Gamma''$ is inconsistent. But it is a finite subset of Γ, and this contradicts the supposition that Γ is finitely consistent. So at least one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\text{‘}\neg\varphi\text{’}\}$ is finitely consistent.

Now suppose that Γ is finitely consistent. We will show that Γ is consistent by constructing a model of Γ. Since the set of sentence letters of PC is denumerable, we can suppose that the sentence letters have been ordered $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$. Define a sequence of sets $\Delta_0, \Delta_1, \Delta_2, \ldots$ recursively as follows: $\Delta_0 = \{\}$; for all $n \geq 0$, $\Delta_{n+1} = \Delta_n \cup \{\Lambda_n\}$ if $\Gamma \cup \Delta_n \cup \{\Lambda_n\}$ is finitely consistent, otherwise $\Delta_{n+1} = \Delta_n \cup \{\text{‘}\neg\Lambda_n\text{’}\}$. Then for each $n \geq 0$ we have: (i) Δ_{n+1} is finite, (ii) $\Gamma \cup \Delta_{n+1}$ is finitely consistent, and (iii) for all $0 \leq m \leq n$ either $\Lambda_m \in \Delta_{n+1}$ or ‘$\neg\Lambda_m$’ $\in \Delta_{n+1}$. Now let I be the following interpretation of PC: Λ_n is true on I iff $\Lambda_n \in \Delta_{n+1}$. Then I is a model of Δ_{n+1}, for all $n \geq 0$. For if $\varphi \in \Delta_{n+1}$ then either $\varphi = \Lambda_k$, for some k, which is true on I by the definition of I, or $\varphi =$ ‘$\neg\Lambda_k$’, for some k, in which case Λ_k is false on I by the definition of I so that φ is true on I. We also have that I is a model of Γ. For take any $\varphi \in \Gamma$. Suppose that φ contains r sentence letters and that they are $\Lambda_{k1}, \ldots, \Lambda_{kr}$. Then for each sentence letter Λ_{ki} in φ, either $\Lambda_{ki} \in \Delta_{kr}$ or ‘$\neg\Lambda_{ki}$’ $\in \Delta_{kr}$. Since $\varphi \cup \Delta_{kr}$ is a finite subset of $\Gamma \cup \Delta_{kr}$, which is finitely consistent, $\varphi \cup \Delta_{kr}$ is consistent. So there is an interpretation on which all the members of Δ_{kr} are true and φ is true. But for each sentence letter Λ_{ki} in φ, either $\Lambda_{ki} \in \Delta_{kr}$ or ‘$\neg\Lambda_{ki}$’ $\in \Delta_{kr}$. So the truth value of the sentence letters in φ is determined by the truth values of the members of Δ_{kr}, and hence the truth value of φ itself is determined by the truth values of the members of Δ_{kr}. So since there is *one* interpretation on which all the members of Δ_{kr} are true and φ is true, on *every* interpretation on which all the members of Δ_{kr} are true φ is true. In particular, φ is true on I. ■

5. Note that entailment in general is not compact:

Premise 1:	John is not a parent of Mary
Premise 2:	John is not a parent of a parent of Mary
Premise 3:	John is not a parent of a parent of a parent of Mary
:	
Conclusion:	Therefore, John is not an ancestor of Mary

The premises of this argument entail the conclusion, but no finite subset of the premises entails the conclusion.

This shows that the argument cannot be expressed in PC (more about this later).

6. Exercises

   a. Prove that (1a), (1b) and (1c) are equivalent.

   b. Prove that for any sets of wffs Γ and Δ (possibly infinite) if $\Gamma \cup \Delta$ is inconsistent then there exists a wff φ such that $\Gamma \models \varphi$ and $\Delta \models$ '$\neg\varphi$'.

   c. Prove that for any sets of wffs Γ and Δ (possibly infinite) if $\Gamma \cup \Delta$ is inconsistent then there exists a wff φ, containing no sentence letter not occurring both somewhere in Γ and somewhere in Δ, such that $\Gamma \models \varphi$ and $\Delta \models$ '$\neg\varphi$'.

Syntactic entailment

1. If Γ and Δ are sets of wffs (possibly empty, possibly infinite), say that **Γ syntactically entails Δ**, written $\Gamma \vdash \Delta$, iff there is a closed tableau whose root wffs are the wffs in the set $\Gamma \cup \{$'$\neg\varphi$'$: \varphi \in \Delta\}$.

2. We are mostly interested in the special case in which Δ has exactly one member: $\Gamma \vdash \varphi$ just in case there is a closed tableau whose root wffs are the wffs in the set $\Gamma \cup$ '$\neg\varphi$'.

3. We have proved the following results for finite sets Γ (possibly empty):

   Soundness: if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

   Proof. Suppose that $\Gamma \vdash \varphi$. So there is a closed tableau whose root wffs are the wffs in $\Gamma \cup$ '$\neg\varphi$'. So $\Gamma \cup$ '$\neg\varphi$' is inconsistent (from the result proved last week). So Γ, '$\neg\varphi$' $\models$. So $\Gamma \models \varphi$. ∎

   Completeness: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$

   Proof. Suppose that it is not the case that $\Gamma \vdash \varphi$. So it is not the case that there is a closed tableau whose root wffs are the wffs in $\Gamma \cup$ '$\neg\varphi$'. So every tableau whose root wffs are the wffs in $\Gamma \cup$ '$\neg\varphi$' has an open branch. In particular, every fully developed tableau whose root wffs are the wffs in $\Gamma \cup$ '$\neg\varphi$' has an open branch, and this branch is fully developed. So $\Gamma \cup$ '$\neg\varphi$' is consistent. So it is not the case that Γ, '$\neg\varphi$' $\models$. So it is not the case that $\Gamma \models \varphi$. So we have shown that if it is not the case that $\Gamma \vdash \varphi$ then it is not the case that $\Gamma \models \varphi$. Contrapositively, if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$. ∎

4. Do these two results also hold for infinite Γ?

   To allow that, we need to allow that tableau can have infinitely many root wffs. That is no problem: even if Γ has infinitely many wffs they are denumerable. So we can list them vertically upwards, leaving an end of the branch at the bottom, at which point development rules can start to be applied. The only problem here is that we can never in practice write down such a list of wffs.

To allow for fully developed branches, we also need to allow that a branch might extend downwards infinitely. That also seems to be no problem – at each step along the way it will be only finite in length, and so will have a point at which the development rule can be applied. The only problem here, again, is that we can never in practice write down such an infinite branch.

Having allowed these two things, we can see by checking the proofs for the finite case that they extend to the infinite case. So yes, these two results also hold for infinite Γ.

5. Semantic entailment and syntactic entailment are both relations between sets of wffs. What the soundness and completeness results tell us is that these two relations have the same **extension** – the set of pairs of wffs between which the first relation holds is identical to the set of pairs of wffs between which the second relation obtains.

   That is: $\{<\Gamma, \Delta>: \Gamma \models \Delta\} = \{<\Gamma, \Delta>: \Gamma \vdash \Delta\}$

6. This means that any result about the extension of the semantic entailment relation also true of the extension of the syntactic entailment relation. All the results about semantic entailment that we saw in the section ‘Some properties of entailment’ are extensional results, so the corresponding results hold for syntactic entailment. For example:

   a. If $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$
   b. If $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \Delta$
   c. If $\Gamma \vdash \Delta$ and $\Delta \subseteq \Delta'$ then $\Gamma \vdash \Delta'$
   d. If $\varphi \vdash \psi$ and $\psi \vdash \chi$ then $\varphi \vdash \chi$
   e. If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$ then $\Gamma, \Delta \vdash \psi$
   f. $\Gamma \vdash$ ‘$(\varphi \wedge \psi)$’ iff $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$

7. Here is how we can use the soundness and completeness of tableau system to prove these results about syntactic entailment:

   Result: If $\varphi \vdash \psi$ and $\psi \vdash \chi$ then $\varphi \vdash \chi$

   Proof. Suppose that $\varphi \vdash \psi$ and $\psi \vdash \chi$. Then $\varphi \models \psi$ and $\psi \models \chi$ (by soundness). So $\varphi \models \chi$ (by the cut result for semantic entailment). So $\varphi \vdash \chi$ (by completeness). ■

8. We could also prove each result directly, without appealing to soundness and completeness, but in many cases it is much more difficult.