Relations

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1. Relations in General

Definition:

An n-place relation on a domain D is a set of ordered n-tuples of elements of D.

Examples:

- {<1, 2, 3>, <2, 3, 4>} is a 3-place relation on the domain of natural numbers. It has exactly two members, each of which is an ordered 3-tuple.

- {<Paris>, <New York>, <Wagga>} is a [ ]-place relation on the domain of [ ]. It has exactly [ ] members, each of which is an ordered [ ]-tuple. We take it to be the same as the set {Paris, New York, Wagga}.

- {<x, y>: x loves y} is a [ ]-place relation on the domain of [ ]. It has [ ] members, each of which is an ordered [ ]-tuple.

- {<a, b, c, n>: n>2 and $a^n + b^n = c^n$} is a [ ]-place relation on the domain of [ ]. That it has no members is [ ] theorem.

Note:

- There are two common ways of specifying a relation on a domain: (a) listing its members, or (b) giving necessary and sufficient conditions for being a member.

- Relations can be empty (i.e. have no members). If a domain is empty then any relation on that domain must be empty.

- n-place relations are also called n-ary or n-adic relations.

- 2-place relations are also called binary relations. We are most interested in these.

- A 2-tuple is also called a pair.

- The order of elements matters in an ordered n-tuple, but not in a set. The ordered pair <1, 2> is distinct from the ordered pair <2, 1>, but the set {1, 2} is identical to the set {2, 1}.

- Elements can be repeated in an n-tuple, but not in a set. <1, 1, 1> is distinct from <1>, but {1, 1, 1} is identical to {1}.

- Verbs are not relations: verbs are bits of language, whereas relations are not. But it is often useful to think of verbs as *standing for* or *expressing* relations. Eg. It is useful to think of ‘love’ as expressing the relation {<x, y>: x loves y} on the domain of living people, and ‘give’ as expressing the relation {<x, y, z>: x gives y to z} on the domain of all things. We sometimes call the relation that a verb expresses its *extension*.

Questions:

1. How many binary relations are there on a domain of three objects?

2. How many empty relations are there?

2. Translating into the language of predicate calculus

We can translate ‘Briony loves Jagjeet’ into predicate calculus as the formula ‘Lab’, if we interpret ‘a’ as standing for Briony, ‘b’ as standing for Jagjeet, and ‘L’ as standing for the relation {<x, y>: x loves y} on the domain of people (or students at Oxford, or ...). Such a translation preserves truth conditions: ‘Briony loves Jagjeet’ is true iff ‘Lab’ is true.

Questions:

Translate the following into predicate calculus suitably interpreted:

1. Everyone loves Jagjeet.

2. The 3-place relation R is not empty on the domain of people.

3. The binary relations R and S are identical on the domain of dogs.

3. Graphing Binary Relations

The good thing about binary relations is that we can *graph* them: if <a, b> is an element of the relation then draw a dot for a and call it ‘a’, a dot for b and call it ‘b’, and an arrow from dot a to dot b.

Example:

Graph the relation {<1, 1>, <1, 2>, <2, 3>, <2, 2>, <3, 2>, <1, 3>} on the domain {1, 2, 3, 4}.

Terminology:

- There is a *loop* on dot a if there is an arrow from a to a.
- There is a *single arrow* from dot a to dot b if there is an arrow from a to b but no arrow from b to a.
- There is a *double arrow* between dots a and b if there is an arrow from a to b, and an arrow from b to a.
- There is a *broken journey* from dot a to dot c if there is a dot b such that there is an arrow from a to b, and an arrow from b to c.
- There is a *broken journey with a short cut* from dot a to dot c if there is a broken journey from a to c, and an arrow from a to c.

- There is a *broken journey without a short cut* from dot a to dot c if there is a broken journey from a to c, but no arrow from a to c.

Note: dots a, b, and c in these definitions do not have to be distinct. So:

- There is one type of loop.
- There are two types of double arrow.
- There are five types of broken journey, three of which automatically have shortcuts.

4. Classifying Binary Relations

Suppose that R is a binary relation on the domain D.

Reflexivity:

R is *reflexive* iff:

$\forall x Rxx$	i.e. Every dot has a loop
$\neg\exists x \neg Rxx$	i.e. There are no dots without a loop

R is *irreflexive* iff:

$\forall x \neg Rxx$	i.e.
$\neg\exists x Rxx$	i.e.

R is *non-reflexive* iff it is neither:

$[\exists x Rxx \wedge \exists x \neg Rxx]$

i.e.

Symmetry:

R is *symmetric* iff:

$\forall x \forall y[Rxy \rightarrow Ryx]$	i.e. Every arrow is a double arrow
$\neg\exists x \exists y[Rxy \wedge \neg Ryx]$	i.e. There are no single arrows

R is *asymmetric* iff:

$\forall x \forall y[Rxy \rightarrow \neg Ryx]$	i.e.
$\neg\exists x \exists y[Rxy \wedge Ryx]$	i.e.

R is *non-symmetric* iff it is neither:

$[\exists x \exists y[Rxy \wedge Ryx] \wedge \exists x \exists y[Rxy \wedge \neg Ryx]]$

i.e.

Transitivity:

R is *transitive* iff:

$\forall x \forall y \forall z[[Rxy \wedge Ryz] \rightarrow Rxz]]$	i.e. Every broken journey has a shortcut
$\neg\exists x \exists y \exists z[[Rxy \wedge Ryz] \wedge \neg Rxz]$	i.e. There are no broken journeys without a shortcut

R is *intransitive* iff:

$\forall x \forall y \forall z[[Rxy \wedge Ryz] \rightarrow \neg Rxz]]$	i.e.
$\neg\exists x \exists y \exists z[[Rxy \wedge Ryz] \wedge Rxz]$	i.e.

R is *non-transitive* iff it is neither:
$[\exists x\exists y\exists z[[Rxy \land Ryz] \land Rxz] \land \exists x\exists y\exists z[[Rxy \land Ryz] \land \neg Rxz]]$
i.e.

Connectedness:

R is *connected* iff:

$\forall x\forall y[\neg x = y \rightarrow [Rxy \lor Ryx]]$ i.e. Every pair of distinct dots have an arrow between them

$\neg\exists x\exists y[\neg x = y \land [\neg Rxy \land \neg Ryx]]$ i.e. There are no distinct dots without an arrow between them

R is *not connected* iff:

$\exists x\exists y[\neg x = y \land [\neg Rxy \land \neg Ryx]]$ i.e.

Other:

R is an *equivalence relation* iff it is reflexive, symmetric and transitive.

Questions:

1. Classify the following relations:

   (a) {<x, y>: x is a sister of y} on the domain of living people.

   (b) {<x, y>: y = x + 2} on the domain of natural numbers.

   (c) {<x, y>: x supports the same football team as y} on the domain of living people.

2. Is it possible to have a non-empty binary relation on a given domain which is both transitive and intransitive?

4. Give a relation on a specified domain that is:

   (a) Irreflexive, non-symmetric, non-transitive, and not connected.

   (b) Reflexive, asymmetric, intransitive, and connected.

5. (a) Use a tableau to show that a transitive relation is irreflexive only if it is asymmetric.

   (b) Two binary relations R and S are said to be converse iff $\forall x\forall y[Rxy \leftrightarrow Syx]$. Show that if a relation is symmetric then so is its converse.